# Recap: What you need to know about presentable stuff

**General thry**: classes of diagram shapes $A, B$ ("small", "filtered") $\rightsquigarrow$ $\left\{\begin{array}{l} A \,\&\, B\text{-colim "generate" small colim} \\ A^{op}\text{-lim} \,\&\, B\text{-colim commute in Ani (or your enriching cat)} \end{array}\right\} \rightsquigarrow$ nice thry of $A$-cocompletion (of $B$-cocomplete cat $\rightsquigarrow$ small ccplt)

| $(A, B) =$ | $\left(\begin{smallmatrix}\phi \\ \text{or} \\ \text{idem}\end{smallmatrix}, \text{all}\right)$ | $(\phi, \amalg, \text{sift})$ | $(\text{fin}, \text{filt})$ | $(\kappa\text{-small}, \kappa\text{-filt})$ |
|---|---|---|---|---|
| locally small cocomplete cats are | atomically gen $\Longrightarrow$ | cpt proj gen $\Longrightarrow$ | cpt gen $\Big(\Rightarrow$ | $\kappa$-cpt gen $\forall \kappa$: inf reg$\Big)$ |

(gen = closure of "cpt" obj under small colim ("weak" generation is enough!))

(warning: $\kappa < \tau$ does not imply $\kappa$-cg $\Rightarrow$ $\tau$-cg but it is true for cofinally many $\kappa < \tau$)

| | | | | |
|---|---|---|---|---|
| of the form: | $\mathcal{C} = \mathcal{P}(\mathcal{C}_0)$ | $\mathcal{P}_\Sigma(\mathcal{C}_0)$ | $\mathrm{Ind}(\mathcal{C}_0)$ | $\mathrm{Ind}_\kappa(\mathcal{C}_0)$ |
| Can take $\mathcal{C}_0 =$ | $\mathcal{C}^{atom}$ | $\mathcal{C}^{cp}$ | $\mathcal{C}^{\aleph_0}$ | $\mathcal{C}^{\kappa}$ |
| $\mathcal{C}$: cocomp translates to: | this choice is automatically idem. cplt. | $\mathcal{C}_0$ has fin coprod | $\mathcal{C}_0$: fin cocomp | $\mathcal{C}_0$: $\kappa$-sm ccpl. |
| Any obj of $\mathcal{C}$ is canonically a $\boxed{\;}$ colim of $\mathcal{C}_0$ | any small | sifted | filtered | $\kappa$-filt. |

$\leftarrow$ strong generation!

(fact: any obj of $\mathcal{P}_\Sigma(\mathcal{C}_0)$ is canonically a geom. real. of $\mathrm{Ind}(\mathcal{C}_0)$)

non-full subcats:

$$\underset{\text{atomic pres functor}}{\mathrm{Pr}^{L,\text{atgen}}} \subset \underset{\text{cp-pres. funct}}{\mathrm{Pr}^{L,\text{cp-gen}}} \subset \underset{\text{cpt obj pres funct}}{\mathrm{Pr}^{L}_{\aleph_0}} \subset \underset{\kappa\text{-cpt obj pres funct.}}{\mathrm{Pr}^{L}_{\kappa}} \subset \mathrm{Pr}^{L} = \bigcup_\kappa \mathrm{Pr}^L_\kappa \subset \widehat{\mathrm{Cat}}$$

(colim-pres)

$$\mathcal{P} \dashv \int \qquad \mathcal{P}_\Sigma \dashv \int \qquad \mathrm{Ind} \dashv \int \qquad \mathrm{Ind}_\kappa \dashv \int$$

$$\mathrm{Cat} \supset \mathrm{Cat}^{idem} \simeq \qquad \underset{\amalg\text{-pres funct.}}{\mathrm{Cat}^{\amalg}} \supset \mathrm{Cat}^{\amalg, idem} \simeq \qquad \underset{\text{rex funct}}{\mathrm{Cat}^{rex}} \supset \mathrm{Cat}^{rex, idem} \simeq \qquad \underset{\kappa\text{-rex funct.}}{\mathrm{Cat}^{\kappa\text{-rex}}}$$

exer
$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$
$L$ pres $\kappa$-cpt obj $\Longleftrightarrow$ $R$ pres $\kappa$-filt colim

$\mathrm{Pr}^R_{(\kappa)}$: presentable cats & right adjoints (pres. $\kappa$-filt colim)

$\mathrm{Pr}^R_\kappa \simeq \mathrm{Pr}^{L, op}_\kappa$

ex $\mathcal{C}_0 = \{R^{\oplus n} \mid n \geq 0\} \subset \mathrm{Mod}^{\heartsuit}_R \rightsquigarrow \underset{\text{proj}}{\mathcal{C}_0^{idem}} \hookrightarrow \underset{\text{flat}}{\mathrm{Ind}(\mathcal{C}_0)}$ ($\mathcal{C}_0$: fin free)

$\mathrm{Ind}(\mathcal{C}_0) \to \mathrm{Fun}^{\times}(\mathcal{C}_0, \mathrm{Set}) = \mathrm{Mod}^{\heartsuit}_R$

$\mathrm{Ind}(\mathcal{C}_0) \hookrightarrow \mathcal{P}_\Sigma(\mathcal{C}_0) = \mathcal{D}_{\geq 0}(R)$

**Prop** $\mathcal{C}$: presentable. $\mathcal{C}_0 \hookrightarrow \mathcal{C}$
- dense $\Longleftrightarrow$ $\mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C}_0)$ f.f.
- weakly dense (colim closure of $\mathcal{C}_0$ is $\mathcal{C}$) $\Longleftrightarrow$ conservative.

ex $\Delta \subset \mathrm{Cat}$
$\{\Delta^1\} \subset \mathrm{Cat}$

**Fact** $\mathrm{Pr}^L, \mathrm{Pr}^R_{(\kappa)} \subset \widehat{\mathrm{Cat}}$ creates limits