## § Cocompletion

**Thm** $\mathrm{LFun}(\mathcal{P}(\mathcal{C}),\mathcal{D}) \underset{-\circ よ}{\overset{\mathrm{LKE}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{Fun}(\mathcal{C},\mathcal{D})$ : equiv.

i.e. $\mathcal{P}$ is a local ladj to $\widehat{\mathrm{Cat}}^{\mathrm{cocomp}} \longrightarrow \widehat{\mathrm{Cat}}$
(2-)

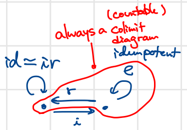
$$\begin{tikzcd} \widehat{\mathrm{Cat}}^{\mathrm{cocomp}} \arrow[r] & \widehat{\mathrm{Cat}} \arrow[d, phantom, "\cup"] \\ \mathrm{at} & \mathrm{Cat} \arrow[llu, bend left, "\mathcal{P}"'] \end{tikzcd}$$

This is a consequence of the following:

**Rmk** representable functors are atomic, i.e. $\mathrm{Map}_{\mathrm{PSh}(\mathcal{C})}(よx, -) : \mathrm{PSh}(\mathcal{C}) \longrightarrow \mathrm{Ani}$ pres. small colimits

**Def** $\mathcal{C}_0 \overset{f}{\hookrightarrow} \mathcal{C}$ is <u>dense</u> if $\mathrm{Lan}_f f = \mathrm{id}$, i.e. $\forall X \in \mathcal{C}$ $\underset{\substack{c \to X \\ c \in \mathcal{C}_0}}{\mathrm{colim}}\, c \xrightarrow{\sim} X$

**ex** $\mathbb{Q} \hookrightarrow \mathbb{R}$ is dense but $\mathbb{Z} \hookrightarrow \mathbb{R}$ is not. $(\mathbb{R} \xrightarrow{\lfloor - \rfloor} \mathbb{R})$



**Thm** $\mathcal{C} \in \mathrm{Cat} \rightsquigarrow \mathcal{C} \xrightarrow{よ} \mathcal{P}(\mathcal{C})$ is dense

**exer** Any atomic obj of $\mathcal{P}(\mathcal{C})$ is a retract of a representable. $\mathcal{C} \hookrightarrow \hat{\mathcal{C}} = \mathcal{P}(\mathcal{C})^{\mathrm{atomic}}$ is the <u>idempotent completion</u>

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, hook] & \mathcal{P}(\mathcal{C}) \arrow[r, "F"'] & \mathcal{D} \\ \mathcal{P}(\mathcal{C}) \arrow[ru, "\mathrm{id}"'] \arrow[rru, bend right, "\text{left Kan extended}"'] & & \\ X & & \end{tikzcd}$$

left Kan extended
⇕
$F$: colim-pres

**Rmk** $\widehat{\mathrm{Cat}}^{\mathrm{cocomp}} \underset{\longrightarrow}{\overset{\overset{\exists}{\longleftarrow}}{\perp}} \widehat{\mathrm{Cat}}$  $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \widehat{\mathrm{Ani}})$ & take the closure. if $\mathcal{C}$: locally small.

$\mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ani}) \subset \mathrm{Fun}(\mathcal{C}^{op}, \widehat{\mathrm{Ani}})$

It is a subcat of $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ani})$. if moreover $\mathcal{C}$: accessible.

**Prop** $\mathcal{C} \hookrightarrow \overset{\text{loc. sm.}}{\mathcal{D}}$ is dense iff the restricted Yoneda emb $\mathcal{D} \longrightarrow \mathcal{P}(\mathcal{D}) \xrightarrow{\mathrm{res}} \mathcal{P}(\mathcal{C})$ is f.f.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "f"'] & \mathcal{P}(\mathcal{C}) \ \mathring{h}_y \arrow[d, shift right, "\mathrm{RKE}_{f^{op}}"'] \\ y \in \mathcal{D} \arrow[r] & \mathcal{P}(\mathcal{D}) \ni h_y \arrow[u, shift right, "-\circ f^{op}"'] \end{tikzcd}$$

$$\begin{tikzcd} \{d\} \arrow[r] & \mathcal{D} \arrow[rd, "\mathrm{id}"] & \\ * \arrow[r] \arrow[u] & \mathcal{C} \arrow[r] \arrow[u] & \mathcal{D} \end{tikzcd}$$

$\mathcal{C}_{/y}^{\wedge} \longrightarrow \mathcal{D}$

$\hookrightarrow$ $\mathcal{D} \underset{\hookrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{P}(\mathcal{C})$ refl. localization

# § Reflective (aka. Bousfield) localization

<u>exer</u> refl. loc ⟺ loc. functor w/ right adj.

<u>Def</u> · $$\begin{tikzcd} \mathcal{C} \arrow[loop below, "L"] \arrow[r, shift left, "f"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "f^R"] \end{tikzcd}$$ is a <u>localization</u> (refl.) if it admits a fully faithful right adjoint.

· A subcategory $\mathcal{D} \subset \mathcal{C}$ is <u>reflective</u> if the incl. admits a left adj.

<u>Prop</u> (localization ⟺ idempotent monad) Let $L : \mathcal{C} \to \mathcal{C}$ be an endofunctor w/ ess. im $L\mathcal{C}$. TFAE:

(1) $\exists$ localization $\mathcal{C} \underset{f^R}{\overset{f}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$ & an equivalence $L \simeq f^R \circ f$

(2) $L : \mathcal{C} \to L\mathcal{C}$ is left adjoint to the inclusion $i : L\mathcal{C} \to \mathcal{C}$

(3) $\exists$ nat. tr $\eta : \mathrm{id}_{\mathcal{C}} \to L : \mathcal{C} \to \mathcal{C}$ s.t. $\forall c \in \mathcal{C}$ $\eta_{Lc}, L(\eta_c) : Lc \to LLc$ are equiv.

<u>proof</u> (1) ⟺ (2) replace $\mathcal{D}$ by its essential image in $\mathcal{C}$

(2) ⟹ (3) $\eta$ : unit of $L \dashv i$

$\rightsquigarrow$ $\forall c_0 \in \mathcal{C}, c_1 \in L\mathcal{C}$ $\mathrm{Map}(Lc_0, c_1) \xrightarrow[\sim]{\eta_{c_0}^*} \mathrm{Map}(c_0, c_1)$

$\hookrightarrow$ · $c_0 = Lc \rightsquigarrow \eta_{Lc} : Lc \xrightarrow{\sim} LLc$ by Yoneda

· $c_0 = c,\ c_1 = LLc$

naturality square $$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "\eta_c"'] \arrow[dr, phantom, "\circlearrowright"] & Lc \arrow[d, "L(\eta_c)"] \\ Lc \arrow[r, "\eta_{Lc}"'] & LLc \end{tikzcd}$$ $\longrightarrow \eta_{Lc} \simeq L(\eta_c)$ : equiv.

(3) ⟹ (2)

enough to show $\eta$ is the unit.

i.e. $\mathrm{Map}_{\mathcal{C}}(Lc_0, Lc_1) \underset{\eta_{c_0}^*}{\xrightarrow{\sim}} \mathrm{Map}_{\mathcal{C}}(c_0, Lc_1)$

· <u>$\pi_0$-surj</u> because $$\begin{tikzcd} c_0 \arrow[r, "\forall f"] \arrow[d, "\eta_{c_0}"'] & Lc_1 \arrow[d, "\eta_{Lc_1}", "\simeq"'] \\ Lc_0 \arrow[r, "Lf"'] \arrow[ur, dashed, "\exists"] & LLc_1 \end{tikzcd}$$

$$\begin{tikzcd} c \arrow[r] \arrow[d] & Lc \arrow[r, "g"] \arrow[d, "\simeq"'] & Ld \arrow[d, "\simeq"] \\ Lc \arrow[r, "\simeq"] \arrow[urr, bend right=60, end anchor=east] & LLc \arrow[r, "Lg"'] \arrow[ur] & LLd \end{tikzcd}$$ $\pi_0$- inj. + map in from $\forall K \in \mathcal{H}$.

· <u>mono</u> $$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(LLc_0, Lc_1) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] \arrow[ddd, bend right=60, "\simeq"'] \arrow[dr, phantom, "\dashv \textcircled{?}"] & \mathrm{Map}_{\mathcal{C}}(Lc_0, Lc_1) \arrow[d] \arrow[ddd, bend left=60, "\simeq"] & \\ \mathrm{Map}_{\mathcal{C}}(Lc_0, Lc_1) \arrow[r] \arrow[d, "\simeq"', "L"] & \mathrm{Map}_{\mathcal{C}}(c_0, Lc_1) \arrow[d, "L"] & \ni f \\ \mathrm{Map}_{\mathcal{C}}(LLc_0, LLc_1) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathcal{C}}(Lc_0, LLc_1) \arrow[d, "\simeq"] & \ni Lf \\ \mathrm{Map}_{\mathcal{C}}(LLc_0, Lc_1) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(Lc_0, Lc_1) & \ni \eta_{Lc_1}^{-1} \circ Lf \end{tikzcd}$$

<u>Prop</u> $$\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "i"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift right, "L"'] \\ K \arrow[u, "f"] \arrow[r] & K^{\triangleright} \arrow[u, "\bar{f}"'] \end{tikzcd}$$

· if $\bar{f}$ : colimit cone, $L \circ \bar{f}$ is a colimit cone of $f$.
(because $L \circ \bar{f}|_K = K$ )

· $\mathcal{C}$ is closed under limits in $\mathcal{D}$

$$\frac{d \xrightarrow{\ if\ } [illegible]}{Ld \longrightarrow f}$$

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{D})}(\underline{d}, if) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & \mathrm{Map}_{\mathcal{D}}(d, \underline{\lim^{\mathcal{D}} if}) \\ \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{Ld}, f) \arrow[r, "\simeq", "i"'] & \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{D})}(\underline{iLd}, if) \arrow[d, phantom, "\parallel"] \\ & \mathrm{Map}_{\mathcal{D}}(iLd, \underline{\lim^{\mathcal{D}} if}) \arrow[uu, bend right=60] \end{tikzcd}$$

<u>$\lim^{\mathcal{D}} if$ is L-local</u>

ess. im of loc. can be characterized by the following:

**Def** Let $W \subset \mathrm{Ar}(\mathcal{C})$. $c \in \mathcal{C}$ is W-local if $\forall f : a \to b$,

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(b, c) \arrow[d, "\simeq"] \\ \mathrm{Map}_{\mathcal{C}}(a, c) \end{tikzcd}$$

**exer** $\begin{tikzcd} \mathcal{D} \arrow[r, hook, shift right, "i"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "L"'] \end{tikzcd}$. $c \in \mathrm{Im}\, i \iff$ local wrt L-equiv. Moreover, $\mathcal{D} \simeq \mathcal{C}[\text{L-equiv}^{-1}]$

**Warning**. $\begin{tikzcd} \{\text{W-loc obj}\} \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, dashed, shift right] \end{tikzcd}$ not nec. exist.

closed under lim

# § Presentable categories, compactness, AFT

idea: presentable cats are cocomplete ($\rightsquigarrow$ usually large) categories that is generated by small set of generators & relations.

| algebra | higher category theory |
|---|---|
| (small) Set | (accessible) $\infty$-category |
| sums | colimits |
| abelian groups | presentable $\infty$-cat |
| free ab gps | presheaves |
| ( products | (finite) limits |
| comm. ring | $\infty$-topos ) |

$$\mathrm{Cat} \xrightarrow{\mathcal{P}} \widehat{\mathrm{Cat}}^{\mathrm{cocompl}} \underset{\text{non-full}}{\subset} \widehat{\mathrm{Cat}}$$

a large cat

**Def** $\mathcal{C}$ is presentable if $\exists\, \mathcal{C}_0$ small, $R \subset \mathcal{P}(\mathcal{C}_0)$ (small) set of morphisms.

$$\mathcal{C} = \mathcal{P}(\mathcal{C}_0 ; R) \underset{\perp}{\overset{\longleftarrow}{\hookrightarrow}} \mathcal{P}(\mathcal{C}_0) \quad \text{full subcat of } R\text{-local obj. i.e.}$$

$$\overset{\cup\!\!\!\!\!|}{X} \iff \forall f : A \to B \in R, \ \mathrm{Map}(B, X) \xrightarrow{\sim} \mathrm{Map}(A, X).$$

$\mathrm{Pr}^L \subset \widehat{\mathrm{Cat}}^{\mathrm{cocompl}}$ full subcat of presentable cats.

**fact** • $\mathcal{C} \subset \mathcal{P}(\mathcal{C}_0)$ closed under lim. $\mathcal{C} \underset{\perp}{\overset{\longleftarrow}{\hookrightarrow}} \mathcal{P}(\mathcal{C}_0) \Rightarrow \mathcal{C}$ is also cocomplete.

•
$$\begin{tikzcd} \coprod_{r \in R} \mathcal{P}(\Delta^1) \arrow[r, "r"] \arrow[d] & \mathcal{P}(\mathcal{C}_0) \arrow[d, shift right] \\ \coprod_{r \in R} \mathcal{P}(\Delta^0) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{P}(\mathcal{C}_0 ; R) \arrow[u, shift right, hook] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

in $\widehat{\mathrm{Cat}}^{\mathrm{cocompl}}$ "presentation"

It turns out that $\mathrm{Pr}^L \subset \widehat{\mathrm{Cat}}^{\mathrm{cocompl}}$ closed under small colim

$\rightsquigarrow$ (it is a closure of presheaves under small colim

& $\mathrm{Pr}^L$ : category of $\Omega$-compact objects

For more effective we need to discuss $\kappa$-filt. colim first. $\kappa$ : infinite (regular) cardinal

$\mathrm{Set}_{<\kappa}$ has $\kappa$-small colim.

**Def** • $K \in \mathrm{sSet}$ is $\kappa$-small if it has finitely many nondeg. simplices.
($\aleph_0$-small = finite)

• $\mathcal{C} \in \widehat{\mathrm{Cat}}$ is $\kappa$-filtered if
($\aleph_0$-small = filtered)

$$\forall \begin{tikzcd} \overset{\kappa\text{-small}}{K} \arrow[r, "f"] \arrow[d] & \mathcal{C} \\ K^{\triangleright} \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$\left( \iff \forall (K, f) \ \ \mathcal{C}_{f/} \neq \phi \right)$

• If $\mathcal{C}$ has $\kappa$-filt colim, $X \in \mathcal{C}$ is $\kappa$-cpt if $\mathrm{Map}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Ani}$ preserves $\kappa$-filt colim. $\longrightarrow \mathcal{C}^{\kappa} \subset \mathcal{C}$ full sub

- For $\mathcal{C}$ small, $\mathrm{Ind}_\kappa(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ full sub of $F : \mathcal{C}^{op} \to \mathrm{An}$ s.t. $\int F = \mathcal{C}_{/F}$ is $\kappa$-filtered. (called the $\kappa$-flat functor)

$\uparrow$

**Rmk** $\forall$ such $F$ $\quad F \simeq \operatorname{colim}_{c \in \mathcal{C}_{/F}} よ c$ filt colim of representables

**Prop** this is closed under $\kappa$-filt colimits [065Z]

**Rem** for $\mathcal{C}$ large, this is the def. $\downarrow$

$\rightsquigarrow$
- the inclusion pres $\kappa$-filt. colim
- $\mathcal{C} \hookrightarrow \mathrm{Ind}_\kappa(\mathcal{C})$ factors through $\kappa$-compact obj.

**Prop** $\forall \mathcal{D}$ with small $\kappa$-filt colim $\quad \mathrm{Fun}^{\kappa\text{-filt}}(\mathrm{Ind}_\kappa \mathcal{C}, \mathcal{D}) \underset{-\circ よ}{\overset{\mathrm{LKE}}{\underset{\sim}{\rightleftarrows}}} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

$$\begin{tikzcd}
\widehat{\mathrm{Cat}}^{\kappa\text{-filt}} \arrow[r, phantom, "\perp"] & \widehat{\mathrm{Cat}} \arrow[l, bend right, "\mathrm{Ind}_\kappa"'] \\
\cup \; {\scriptstyle \text{non-full } \kappa\text{-cpt pres.}} & \cup \\
\mathrm{Acc}_\kappa \arrow[r, bend right, "\mathrm{idem}"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend right] \\
\cup & \cup \\
\mathrm{Pr}_\kappa \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathrm{Cat}^{\kappa\text{-rex}} \arrow[l, bend right]
\end{tikzcd}$$
$$\begin{tikzcd}
\widehat{\mathrm{Cat}}^{\kappa\text{-filt}} \arrow[r, "\mathrm{forget}"'] & \widehat{\mathrm{Cat}}
\end{tikzcd}$$

**Prop** Consider $\mathcal{C} \overset{f}{\hookrightarrow} \mathcal{D}$ fully faithful, where $\mathcal{D}$ has $\kappa$-filt colim

$$\begin{tikzcd}
\rightsquigarrow & \mathcal{C} \arrow[r, hook, "f"] \arrow[d] & \mathcal{D} \\
& \mathrm{Ind}_\kappa(\mathcal{C}) \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

(1) if $\mathcal{C} \subset \mathcal{D}^\kappa$, $\bar{f}$ is fully faithful

(2) if moreover $\mathcal{C}$ generates $\mathcal{D}$ under $\kappa$-filt colim (i.e. the $\kappa$-filt colim closure of $\mathcal{C}$ is $\mathcal{D}$) then $\bar{f}$ : equiv.

$X = \operatorname{colim} よ x_i$

**Proof** exer. (harder if $\mathcal{C}$: large) (Prop 1: reduce to when $\mathcal{D} = \mathrm{An}^{op}$ & use $\mathcal{P}(\mathcal{C})$)

ex many large cats we know, also flat modules $\quad$ rmk $\mathcal{C} : (n,1)\text{-cat} \Rightarrow \mathrm{Ind}_\kappa(\mathcal{C}) : (n,1)\text{-cat}$

**Def** $\mathcal{C}$ : ($\kappa$-)accessible $\iff \exists \kappa$ $\exists \mathcal{C}_0$ small, $\mathrm{Ind}_\kappa(\mathcal{C}_0) \xrightarrow{\sim} \mathcal{C}$ ($\hookrightarrow$ can take $\mathcal{C}_0 = \mathcal{C}^\kappa$)

A functor $f : \mathcal{C} \to \mathcal{D}$ is accessible if it is $\kappa$-conti for some $\kappa$

($\hookrightarrow$ if $\mathcal{C}, \mathcal{D}$ : acc, $\exists \tau$ $f : \tau$-conti & pres. $\tau$-cpt obj (HTT Rmk 5.4.2.13))

warning $\kappa$-acc does not imply $\tau$-acc for $\tau > \kappa$. but it does if $\tau \gg \kappa$. i.e. if $\forall \kappa_0 < \kappa$, $\tau_0 < \tau$ $\quad \tau_0^{\kappa_0} < \tau$

$\hookrightarrow$ If $\mathcal{C}$ acc, $\mathcal{C}^\tau$ : small $\forall \tau$ & $\bigcup_\tau \mathcal{C}^\tau = \mathcal{C}$

$\mathrm{Acc}_\kappa$ : $\kappa$-acc functors & $\kappa$-conti $\kappa$-cpt pres functors

$$\begin{tikzcd}
\mathrm{Acc}_\kappa \arrow[r, bend right, "(-)^\kappa"'] \arrow[dr, "\simeq"'] & \mathrm{Cat} \arrow[l, bend right, "\mathrm{Ind}_\kappa"'] \\
& \mathrm{Cat}^{\mathrm{idem}} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

**Rmk**
- Small cats are accessible iff idem cplt.
- $\mathrm{Acc} = \bigcup \mathrm{Acc}_\kappa \hookrightarrow \widehat{\mathrm{Cat}}$ creates limits (in fact, Cat-weighted lim)
- adjoints between accessible cats are accessible.

**Def** $\mathcal{C}$ : $\kappa$-compactly generated $\iff$ cocomplete & $\kappa$-accessible,

$\overset{☆}{\iff} \mathrm{Ind}_\kappa(\mathcal{C}_0) \xrightarrow{\sim} \mathcal{C}$ for $\mathcal{C}_0$ : small with $\kappa$-small colim.

$\iff \kappa$-conti localization of $\mathcal{P}(\mathcal{C}_0)$

$$\begin{tikzcd}
& \operatorname{colim} よ c & \text{“}\operatorname{colim}\text{” } よ c \; {\scriptstyle \kappa\text{-sm.}} \arrow[l, mapsto] \\
\rightsquigarrow & \mathcal{C}_0 \arrow[r, bend right] \arrow[r, phantom, "\perp"] \arrow[d] & \mathcal{P}(\mathcal{C}_0)^\kappa \arrow[l, bend right, "\exists"'] \arrow[d] \\
& \mathrm{Ind}\,\mathcal{C}_0 \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathrm{Ind}_\kappa(\mathcal{P}(\mathcal{C}_0)^\kappa) = \mathcal{P}(\mathcal{C}_0) \arrow[l, bend right]
\end{tikzcd}$$

$\mathrm{Pr}^L_\kappa$

$\cap$

$\mathrm{Pr}^L$

**Prop** presentable $\iff \exists$ acc. loc. of $\mathcal{P}(\mathcal{C}_0) \iff$ cocomplete & accessible.

☆ • $$\left.\begin{matrix}\text{A category has}\\ \text{a functor preserves}\end{matrix}\right\} \left\{\begin{matrix}\text{small}\\ \text{finite}\end{matrix}\right\} \text{colimits iff it } \begin{matrix}\text{has}\\ \text{preserves}\end{matrix} \left\{\begin{matrix}\kappa\text{-small} + \kappa\text{-filt}\\ \text{final \& pb} \quad \text{or} \quad \text{fin. coprod \& coeq.}\end{matrix}\right.$$

• $I : \kappa$-filtered $\iff$ $\mathrm{colim} : \mathrm{Fun}(I, \mathrm{Ani}) \longrightarrow \mathrm{Ani}$ pres. $\kappa$-small <u>limits</u>

<u>Cor</u> $\kappa$-compact objects are closed under $\kappa$-small colimits

☺ $$\mathrm{Map}_{\mathcal{C}}(\underset{\underset{\kappa\text{-small}}{\uparrow}}{\underset{I}{\mathrm{colim}}}\, X_i\, ,\, \underset{\underset{\kappa\text{-filt}}{\uparrow}}{\underset{J}{\mathrm{colim}}}\, Y_j) \simeq \lim_I \mathrm{Map}_{\mathcal{C}}(X_i, \mathrm{colim}_J Y_j)$$
$$\simeq \lim_I \mathrm{colim}_J \mathrm{Map}_{\mathcal{C}}(X_i, Y_j)$$
$$\xleftarrow{\sim} \mathrm{colim}_J \lim_I \mathrm{Map}_{\mathcal{C}}(X_i, Y_j)$$
$$\simeq \mathrm{colim}_J \mathrm{Map}_{\mathcal{C}}(\mathrm{colim}_I X_i, Y_j)$$

<u>Adjoint functor thm:</u> (1) $\mathcal{C} \in \mathrm{Pr}^L$, $\mathcal{D}$: loc.sm $\Rightarrow$ $\left(\mathcal{C} \xrightarrow{f} \mathcal{D} \text{ ladj} \iff \text{pres. small colim}\right)$

(2) $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^L$ $\Rightarrow$ $\left(f : \mathcal{C} \xrightarrow{f} \mathcal{D} \text{ radj} \iff \text{acc. \& pres. small colim}\right)$

<u>ex</u> if $\mathcal{C}$ presentable, $W \subset \mathrm{Ar}(\mathcal{C})$ <u>small</u>

$$\rightsquigarrow \quad \begin{tikzcd} \{W\text{-loc. obj}\} \arrow[r, hook, shift right, ""' name=a] & \mathcal{C} \arrow[l, shift right, "\exists L"'] \end{tikzcd}$$

$$\{W\text{-loc. obj}\} \underset{\hookrightarrow}{\overset{\exists L}{\underset{\perp}{\leftarrow}}} \mathcal{C}$$

(In fact, $\forall$ acc. loc is of this form)

<u>Def</u> $\mathrm{Pr}^R \subset \widehat{\mathrm{Cat}}$ presentable cats & right adjoints.

$\mathrm{RFun}(\mathcal{C}^{op}, \mathcal{D})$

<u>Prop</u> $\left.\begin{matrix}\mathrm{Pr}^L \subset \widehat{\mathrm{Cat}}\\ \mathrm{Pr}^R \subset \widehat{\mathrm{Cat}}\end{matrix}\right\}$ creates small colimits

idea: the limit of $S \to \mathrm{Pr}^{L/R} \to \widehat{\mathrm{Cat}}$
= Cartesian sections of some fib w/ presentable fibers.

$\mathrm{Pr}^L_\kappa$

<u>exer?</u> $$\mathrm{LFun}(\mathrm{RFun}(\mathcal{C}^{op}, \mathcal{D}), \mathcal{E}) \simeq \mathrm{LFun}(\mathcal{C}, \mathrm{LFun}(\mathcal{D}, \mathcal{E}))$$

$$\begin{array}{l} \mathrm{RFun}(\mathcal{C}^{op}, \mathcal{D}) \\ \quad \wr\wr \\ \mathrm{LFun}(\mathcal{C}, \mathcal{D}^{op})^{op} \\ \quad \wr\wr \longleftarrow \text{AFT} \\ \mathrm{RFun}(\mathcal{D}^{op}, \mathcal{C}) \end{array}$$

ex · $\mathcal{C}$ with final obj. is $\kappa$-filt $\forall \kappa$

· $X \in \mathrm{Set}$ is $\kappa$-cpt $\Leftrightarrow$ $\# X < \kappa$.

$\mathcal{C}$: filtered $\Rightarrow$ $|\mathcal{C}| = *$

😕

$$\begin{tikzcd}
\overset{\text{finite}}{\partial\Delta^n} \arrow[rr, bend left] \arrow[r] & \overset{\exists}{|K|} \arrow[r] \arrow[d] & {|\mathcal{C}|} \\
& {|K^{\triangleright}|} \arrow[ur, "\exists \text{ by } \mathcal{C}\text{: filtered}"'] &
\end{tikzcd}$$

$\overset{\text{as qcat}}{|\mathcal{C}|} \simeq \underset{\substack{\text{filt. colim} \\ \text{along mono}}}{\operatorname{colim}} |\overset{\text{finite sset}}{K}|$

$\rightsquigarrow$ also a $\overset{\text{filt.}}{\text{colim}}$ in Cat

<u>Prop</u> TFAE: (1) $\mathcal{C}$: $\kappa$-filtered $\quad (\forall f,\ \mathcal{C}_{f/} \neq \phi)$

(2) $\forall K \xrightarrow{f} \mathcal{C}$ $\quad K$: $\kappa$-sm. $\Rightarrow$ $\mathcal{C}_{f/}$ : $\kappa$-filtered

(3) $\forall K$ : $\kappa$-sm, the diagonal $\mathcal{C} \longrightarrow \mathrm{Fun}(K, \mathcal{C})$ is cofinal. $\left( \underset{\text{Q.Th-A}}{\Leftrightarrow} |\mathcal{C}_{f/}| \simeq * \right)$

😕

(1) $\Rightarrow$ (2) join of $\kappa$-small ssets is $\kappa$-small. $\qquad$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) clear

<u>Fact</u> · $\forall \mathcal{C}$ : $\infty$-cat, $\exists$ cofinal map from a poset $P \longrightarrow \mathcal{C}$

If $\mathcal{C}$: filtered, $\exists P$: $\kappa$-filtered ($\kappa = \aleph_0 \rightsquigarrow$ directed in classical sense)

· If $\kappa > \aleph_0$, $\mathcal{C} \in \mathrm{qCat}$ or Kan is $\kappa$-small (as sset)

$\Leftrightarrow$ $\mathcal{C}$ is $\kappa$-compact $\in \mathrm{Cat}$ (or Ani). This is false for $\kappa = \aleph_0$.

$\mathrm{Ani}^{\aleph_0} \supsetneq \mathrm{Ani}^{\mathrm{fin}}$

## § sifted colimits. projective objects. animation

**Def** A simplicial set $K$ is sifted $\iff$ $K \neq \phi$, $\Delta : K \to K \times K$ is cofinal

$\iff \forall I$: fin set. $\Delta : K \to K^I$ is cofinal.

$\uparrow$

**Rem** invariant under Cat. eq.

$\rightsquigarrow$ property of an $\infty$-cat.

$\llcorner$ **Prop** cofinal $\Rightarrow$ w.ho. eq.

$K \xrightarrow[\Delta]{} K \times K$ : w.h.e. $\Rightarrow |K| \simeq *$

**Quillen's thm A** $\rightsquigarrow$ $\mathcal{C}$ : sifted $\iff$ $\mathcal{C} \neq \phi$, $(\mathcal{C}\times\mathcal{C})_{(x,y)/} \underset{\mathcal{C}\times\mathcal{C}}{\times} \mathcal{C}$ : weakly contractible

$\simeq \mathcal{C}_{(x \sqcup y)/}$ if $\mathcal{C}$ has $\sqcup$

$\wr\wr$

$(\mathcal{C}_{x/} \times \mathcal{C}_{y/}) \underset{\mathcal{C}\times\mathcal{C}}{\times} \mathcal{C} \simeq \mathcal{C}_{x/} \underset{\mathcal{C}}{\times} \mathcal{C}_{y/})$

**Important examples:**
- filtered $\Rightarrow$ sifted
- $\Delta^{op}$ is sifted.

**proof** NTS

$$\Delta^{op}_{[m]/} \underset{\Delta^{op}}{\times} \Delta^{op}_{[n]/} \simeq \left(\Delta_{/[m]} \underset{\Delta}{\times} \Delta_{/[n]}\right)^{op} \text{ : weakly contr.}$$

$\wr\wr \leftarrow$ use $\Delta \hookrightarrow \mathrm{Cat}$

$\Delta_{/[m]\times[n]}$

I always get confused how to directly show

$A \underset{C}{\times} B \simeq (A \times B) \underset{C\times C}{\times} C$

So I record it here :

$$\begin{tikzcd}
 & A\times B \arrow[rr] \arrow[dd] & & C\times B \arrow[dd] \\
A\underset{C}{\times}B \arrow[ur] \arrow[rr] \arrow[dd] & & B \arrow[ur] \arrow[dd] & \\
 & A\times C \arrow[rr, dashed] & & C\times C \\
A \arrow[ur] \arrow[rr] & & C \arrow[ur] &
\end{tikzcd}$$

back. front are pb

right is pb bc

$$\begin{tikzcd}
B \arrow[r, "{(g,id)}"'] \arrow[d] & C\times B \arrow[r, "pr_2"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\
C \arrow[r, "\Delta"'] & C\times C \arrow[r, "pr_2"'] & C
\end{tikzcd}$$

Similarly for bottom.

$\rightsquigarrow$ so is top. & left

**Note**

$$\begin{tikzcd}
\mathrm{Cat} \arrow[r, "|-|"] & \mathrm{Ani} \\
\Delta \arrow[u, hook] \arrow[ur, "\mathrm{const}_*"'] &
\end{tikzcd}$$

$|-|$ is left-Kan extended from $\Delta$

$(\mathrm{LFun}(\mathrm{Cat}, \mathrm{Ani}) \hookrightarrow \mathrm{LFun}(\mathrm{PSh}(\Delta), \mathrm{Ani}) \simeq \mathrm{Fun}(\Delta, \mathrm{Ani}))$

In particular. $\forall X \in \mathrm{Cat}$, $|X| \simeq \underset{\Delta/X}{\mathrm{colim}} * \simeq |\Delta_{/X}|$

Kerodon does more complicated argument. hope I'm not missing something...

So $|\Delta_{/[m]\times[n]}| \simeq |[m]\times[n]| \simeq *$. $\boxed{q}$

$\uparrow$ has terminal obj

**Fact**
- finite product & sifted colim commute in Ani
- sifted colimit "=" filt. colim + geom real.
- sifted colim + fin coprod "=" all colim

$\rightsquigarrow$ **Def.** Compact projective
- compact proj gen.
- $\exists \mathcal{C}_0$ $P_\Sigma(\mathcal{C}_0) \xrightarrow{\sim} \mathcal{C}$

$\wr$

$\mathcal{C}_0^{op} \to \mathrm{Ani}$ fin prod preserving

Analogous theory for 1-sifted colim: replace geom.

$$\mathcal{C} : 1\text{-cp gen} \Rightarrow \mathcal{C}^{1\text{-cp}} \Rightarrow P_\Sigma(\mathcal{C}^{1\text{-cp}}) \quad \underline{\text{animation of } \mathcal{C}}.$$

$\uparrow$ localization

$\mathrm{Fun}(\Delta^{op}, \mathcal{C})$

# Recap: What you need to know about presentable stuff

**General thry**: classes of diagram shapes $A, B$ ("small", "filtered") $\rightsquigarrow$ $\left\{ \begin{array}{l} A \,\&\, B\text{-colim "generate" small colim} \\ A^{op}\text{-lim} \,\&\, B\text{-colim commute in Ani (or your enriching cat)} \end{array} \right\}$ $\rightsquigarrow$ nice thry of $A$-cocompletion (of $B$-cocomplete cat $\rightsquigarrow$ small ccplt)

$(A, B) = \left( \begin{array}{c} \phi \\ \text{or} \\ \text{idem} \end{array}, \text{all} \right)$ $\quad (\phi, \amalg, \text{sift}) \quad (\text{fin}, \text{filt}) \quad (\kappa\text{-small}, \kappa\text{-filt})$

locally small cocomplete cats are (gen = closure of "cpt" obj under small colim ("weak" generation is enough!))

atomically gen $\Longrightarrow$ cpt proj gen $\Longrightarrow$ cpt gen ($\Longrightarrow$ $\kappa$-cpt gen $\forall \kappa$: inf reg)

(warning: $\kappa < \tau$ does not imply $\kappa$-cg $\Rightarrow$ $\tau$-cg but it is true for cofinally many $\kappa < \tau$)

of the form: $\mathcal{C} = \mathcal{P}(\mathcal{C}_0)$ $\quad \mathcal{P}_\Sigma(\mathcal{C}_0) \quad \mathrm{Ind}(\mathcal{C}_0) \quad \mathrm{Ind}_\kappa(\mathcal{C}_0)$

Can take $\mathcal{C}_0 =$ $\quad \mathcal{C}^{\text{atom}} \quad \mathcal{C}^{cp} \quad \mathcal{C}^{\aleph_0} \quad \mathcal{C}^{\kappa}$

this choice is automatically idem. cplt.

$\mathcal{C}$: cocomp translates to: $\quad$ — $\quad \mathcal{C}_0$ has fin coprod $\quad \mathcal{C}_0$: fin cocomp $\quad \mathcal{C}_0$: $\kappa$-sm ccpl.

Any obj of $\mathcal{C}$ is canonically a [illegible] colim of $\mathcal{C}_0$: $\quad$ any small $\quad$ sifted $\quad$ filtered $\quad$ $\kappa$-filt. $\quad$ ← strong generation!

(fact: any obj of $\mathcal{P}_\Sigma(\mathcal{C}_0)$ is canonically a geom. real. of $\mathrm{Ind}(\mathcal{C}_0)$)

non-full subcats: $\mathrm{Pr}^{L,\text{atgen}} \subset \mathrm{Pr}^{L,\text{cp-gen}} \subset \mathrm{Pr}^{L}_{\aleph_0} \subset \mathrm{Pr}^{L}_{\kappa} \subset \mathrm{Pr}^{L} = \bigcup_\kappa \mathrm{Pr}^{L}_{\kappa} \subset \widehat{\mathrm{Cat}}$

atomic pres functor; cp-pres. funct; cpt obj pres funct; $\kappa$-cpt obj pres funct.; colim-pres

$$\begin{tikzcd} \mathrm{Pr}^{L,\text{atgen}} \arrow[d, shift left] \arrow[drr, bend left, "\simeq"] & & \\ \mathrm{Cat} \arrow[u, shift left, "\mathcal{P}"] \arrow[u, phantom, "\dashv"] & \supset & \mathrm{Cat}^{\text{idem}} \end{tikzcd}
\qquad
\begin{tikzcd} \mathrm{Pr}^{L,\text{cp-gen}} \arrow[d, shift left] \arrow[drr, bend left, "\simeq"] & & \\ \mathrm{Cat}^{\amalg} \arrow[u, shift left, "\mathcal{P}_\Sigma"] \arrow[u, phantom, "\dashv"] & \supset & \mathrm{Cat}^{\amalg,\text{idem}} \end{tikzcd}$$

$\amalg$-pres funct.

$$\begin{tikzcd} \mathrm{Pr}^{L}_{\aleph_0} \arrow[d, shift left] \arrow[drr, bend left, "\simeq"] & & \\ \mathrm{Cat}^{\text{rex}} \arrow[u, shift left, "\mathrm{Ind}"] \arrow[u, phantom, "\dashv"] & \supset & \mathrm{Cat}^{\text{rex,idem}} \end{tikzcd}
\qquad
\begin{tikzcd} \mathrm{Pr}^{L}_{\kappa} \arrow[d, shift left, "\simeq"] \\ \mathrm{Cat}^{\kappa\text{-rex}} \arrow[u, shift left, "\mathrm{Ind}_\kappa"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

rex funct $\quad$ $\kappa$-rex funct.

(**exer**

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

$L$ pres $\kappa$-cpt obj $\Longleftrightarrow$ $R$ pres $\kappa$-filt colim

$\mathrm{Pr}^{R}_{(\kappa)}$: presentable cats & right adjoints (pres. $\kappa$-filt colim)

$\mathrm{Pr}^{R}_{\kappa} \simeq \mathrm{Pr}^{L,op}_{\kappa}$

**ex** $\mathcal{C}_0 = \{R^{\oplus n} \mid n \geq 0\} \subset \mathrm{Mod}_R^{\heartsuit} \rightsquigarrow \mathcal{C}_0^{\text{idem}} \hookrightarrow \mathrm{Ind}(\mathcal{C}_0)$

$\mathcal{C}_0$: fin free $\quad \mathcal{C}_0^{\text{idem}} =$ proj $\quad \mathrm{Ind}(\mathcal{C}_0) =$ flat

$\mathrm{Ind}(\mathcal{C}_0) \hookrightarrow \mathrm{Fun}^{\times}(\mathcal{C}_0, \mathrm{Set}) = \mathrm{Mod}_R^{\heartsuit}$

$\mathrm{Ind}(\mathcal{C}_0) \hookrightarrow \mathcal{P}_\Sigma(\mathcal{C}_0) = \mathcal{D}_{\geq 0}(R)$

**Prop** $\mathcal{C}$: presentable. $\mathcal{C}_0 \hookrightarrow \mathcal{C}$

- dense $\Longleftrightarrow$ $\mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C}_0)$ f.f.
- weakly dense (colim closure of $\mathcal{C}_0$ is $\mathcal{C}$) $\Longleftrightarrow$ conservative.

**ex** $\Delta \subset \mathrm{Cat}$

$\{\Delta^1\} \subset \mathrm{Cat}$

**Fact** $\mathrm{Pr}^{L}, \mathrm{Pr}^{R}_{(\kappa)} \subset \widehat{\mathrm{Cat}}$ creates limits

## Part 2. higher algebra (rings & modules)

$$\begin{tikzcd}
\text{Set} \arrow[r, squiggly] \arrow[d, squiggly] & \text{monoid} \arrow[d, squiggly] & {} & {} \arrow[l, squiggly] & \text{Category} & \text{(sym) mon.cat} \arrow[l] & {} \arrow[r, squiggly, "\text{``derive''}"] & {} & \infty\text{-category} & \text{(sym) mon } \infty\text{-Cat} \arrow[l] \\
\text{ab.gps} \arrow[r, squiggly] & \text{ring} & & & \text{abelian cats} \arrow[u] & \text{abelian mon cat} \arrow[l] \arrow[u] & & & \text{stable } \infty\text{-Cat} \arrow[u] & \text{stable mon } \infty\text{-cat.} \arrow[l, squiggly] \arrow[u]
\end{tikzcd}$$

(linear alg)

## § Stable categories

**Def** • A category $\mathscr{C}$ is pointed if $\exists\, 0 \in \mathscr{C}$ : zero obj i.e. the object which is both initial & terminal.

($\Longleftrightarrow$ $\exists$ initial $\phi$ & terminal $*$ obj. and the canonical map $\phi \to *$ is an iso.)

• A category $\mathscr{C}$ is stable if
(1) $0 \in \mathscr{C}$.
(2) finite limits & colimits exist. ($\overset{\exists 0}{\Leftarrow}$ pushouts & pullbacks)
(3) A square $\Delta^1 \times \Delta^1 \to \mathscr{C}$ is cartesian iff cocartesian

**Slogan** Stability = fin lim & fin colim "agree".

**Def** • $F : \mathscr{C} \to \mathscr{D}$ between cats w/ fin. lim is left exact if $F$ pres. fin lim.
• colim right exact colim
• exact if both.

exer ↑ all equivalent if $\mathscr{C}, \mathscr{D}$ : stable.  $Cat^{st} \subset Cat$ non-full of stable cats & exact functors

**Rmk** • $Cat^{st} \subset Cat$ creates $\left\{ \begin{array}{l} \text{filt colim.} \\ Fun(K,-) \text{ \& small limits. (i.e. Cat-weighted lim)} \end{array} \right.$
• $Ind_K(-)$, $(-)^K$ of stable categories are stable.

$$\begin{tikzcd}
Cat^{st} \arrow[r, shift left, "Ind"] & Pr^L_{\omega, st} \arrow[l, hook, shift left, "(-)^{\omega}"] \arrow[dl, "\simeq"] \\
Cat^{perf} \arrow[u, phantom, "\cup" description] &
\end{tikzcd}$$

$\rightsquigarrow$ •

• $$\begin{tikzcd}
Cat^{st, K\text{-rex}} \arrow[r, shift left, "Ind_K"] \arrow[r, phantom, "\simeq" description] & Pr^L_{K, st} \arrow[l, hook, shift left, "(-)^K"]
\end{tikzcd}$$

**Rem**
I think $Cat^{st} \underset{\hookrightarrow}{\overset{I}{\leftarrow}} Cat$ is given by the stable closure of Sp-Yoneda emb $\mathscr{C} \hookrightarrow Fun(\mathscr{C}^{op}, Sp)$.

**Prop** • $\mathscr{C}$ with pushouts.

$$\begin{tikzcd}
 & & \{\text{po sq}\} \arrow[d, phantom, "\cap"] & & \\
\rightsquigarrow & \mathrm{Fun}(\Lambda^2_0, \mathscr{C}) \arrow[ur, "\sim"] \arrow[r, shift left, "\mathrm{LKE}"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}([1]^2, \mathscr{C}) \arrow[l, shift left, "\mathrm{res}"] \arrow[r, shift left, "\mathrm{res}"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\Lambda^2_2, \mathscr{C}) \arrow[l, shift left, dashed] \arrow[dl, "\sim"] & \\
 & & \{\text{pb sq}\} \arrow[u, phantom, "\cup"] & &
\end{tikzcd}$$

- $\mathrm{res} \circ \mathrm{LKE}$ has a radj iff $\mathscr{C}$ has pullbacks & the radj is given by pullback

  po = pb iff this is an adjoint equiv iff $\mathrm{res} \circ \mathrm{LKE}$ is an equivalence.

**Next goal**: weaken (2) & (3)

**Def** $\forall x, y \in \mathscr{C}$. $* \simeq \mathrm{Map}(x, 0) \times \mathrm{Map}(0, y) \xrightarrow{\circ} \mathrm{Map}(x, y)$ picks up a morphism $x \to y$.
denoted by $0$.

$\mathscr{C} = \mathrm{Ani}_* \rightsquigarrow X \xrightarrow{0} Y$ is a constant map at the base point.

⚠ zero morphism can have a nontrivial automorphism; e.g. $\forall X$: conn. $(S^0 \longrightarrow X) \simeq (S^0 \xrightarrow{0} X)$.

($\Omega X$ many identifications)

**Def** • A *triangle* in $\mathscr{C}$ is a commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\
0 \arrow[r] & C
\end{tikzcd}$$

often denoted by $A \xrightarrow{f} B \xrightarrow{g} C$ by abuse of notation, making the homotopy $g \circ f \simeq 0$ implicit.
(but it is a part of data)

- $A \xrightarrow{f} B \xrightarrow{g} C$ is a *(co)fiber seq* if the square is (co)cartesian

  $\hookrightarrow$ $A$ (or $f$) is a *fiber* of $g$ $(A \xrightarrow{\sim} \mathrm{fib}(g))$
  $C$ — *cofiber* — $f$ $(\mathrm{cof}(f) \xrightarrow{\sim} C)$

- $$\begin{tikzcd}
A \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & B
\end{tikzcd} \rightsquigarrow \begin{cases} \bullet\ A \simeq \Omega B & \text{if it is cartesian (loop)} \\ \bullet\ \Sigma A \simeq B & \text{if it is cocartesian (suspension)} \end{cases}$$

**exer** $\mathscr{C}$: pointed, has fibers & cofibers / loop & suspension

$\Rightarrow$
- define $\mathrm{Fun}(\Delta^1, \mathscr{C}) \underset{\mathrm{fib}}{\overset{\mathrm{cof}}{\rightleftarrows}} \mathrm{Fun}(\Delta^1, \mathscr{C})$ ($\perp$). It has a radj iff $\mathscr{C}$ has fibers.

  $X \xrightarrow{f} Y \longmapsto Y \to \mathrm{cof}(f)$

  (then radj is given by $(Y \xrightarrow[g]{} Z) \mapsto (\mathrm{fib}(g) \to Y)$)

- Similarly for $\mathscr{C} \underset{\Omega}{\overset{\Sigma}{\rightleftarrows}} \mathscr{C}$ ($\perp$).

When $\mathscr{C}$: stable, these adjunctions are equivalences.

shift notation
$$\begin{cases} \Sigma^n X =: X[n] \\ \Omega^n X =: X[-n] \end{cases}$$

**Prop** For $\mathcal{C}$ pointed,

$\mathcal{C}$: stable $\Longleftrightarrow$ $\begin{cases} (2') \ \forall f: X \to Y \text{ has a fiber (\& a cofiber)} \\ (3') \text{ A triangle } X \xrightarrow{f} Y \xrightarrow{g} Z \text{ is a fib seq iff it is a cofib seq.} \end{cases}$

$\mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow[\sim]{\mathrm{fib}} \mathrm{Fun}(\Delta^1, \mathcal{C})$

$\Longleftrightarrow$ $\begin{cases} (2) \ \mathcal{C} \text{ has finite limits (\& fin. colim.)} \\ (3'') \ (\Sigma \dashv) \Omega \text{ is an equivalence.} \end{cases}$

**proof sketch**
- $(3) \Rightarrow (3') \Rightarrow (3'')$ is clear.
- $(2') + (3'') \Rightarrow$ additive

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[r] \arrow[d] \arrow[drr, phantom, "\ulcorner"] & & 0 \arrow[d] \\ 0 \arrow[r] & X \sqcup Y \arrow[rr] \arrow[dd] \arrow[dr, "\sim"] & & X \arrow[dd] \\ & & X \times Y \arrow[ur] \arrow[dl] & \\ & 0 \arrow[rr] & & \Sigma Y \end{tikzcd}$$

Semiadditive $+ \ \forall X \simeq \underbrace{\Omega\Sigma X}_{\text{group}}$

$\mathrm{Map}(Y, \Omega\Sigma X) \simeq \left( \begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[ur, Rightarrow, "\alpha"] & \Sigma X \end{tikzcd} \right) = \begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[ur, Rightarrow, "\alpha^{-1}"] & \Sigma X \end{tikzcd}$

$\rightsquigarrow \mathrm{eq}(X \underset{g}{\overset{f}{\rightrightarrows}} Y) \simeq \mathrm{fib}(X \xrightarrow{f-g} Y)$. $\Rightarrow (2)$.

- $(3') \Rightarrow (3)$ exer. $\quad (2)(3'') \Rightarrow (3)$ harder exer.

**Def** $S^0 = (*)_+ \in \mathrm{Ani}_*$, $S^n = \Sigma^n S^0 \rightsquigarrow \Omega^n X = \mathrm{Map}_*(S^n, X)$, $\pi_n X := \pi_0 \Omega^n X \in \overset{n \geq 1}{Gp}, \overset{n \geq 2}{Ab}$

from the cogroup str on $S^1 \in \mathrm{ho}(\mathrm{Ani}_*)$

**Rem** fib seq $\Rightarrow$ long exact seq of homotopy groups. (reduces to: $\underset{\text{preserves fibers}}{\pi_0 : \mathrm{Ani}_* \to \mathrm{Set}_*}$ )