§1 Introduction

(n,k)-category … a version of higher category when we consider
$0 \le k \le n$ $\quad 0, \dots, n$ - morphisms
& $> k$ - mor are all invertible.

ex
- $(0,0)$-cat $=$ Set
- $(1,1)$-cat $=$ Cat
- $(1,0)$-cat $=$ gpd
- $(2,1)$-cat $=$ "bicategory"

enrich
enrich
$(+1,+1)$

ex $\mathcal{C}$ with products.
$\rightsquigarrow B\mathcal{C}$
- obj $= *$
- mor $= \mathrm{ob}\,\mathcal{C}$
- $* \xrightarrow[c]{} * \xrightarrow[d]{} * \;=\; * \xrightarrow[\underbrace{c \times d}]{} *$

only defined up to can. isom.

strict vs weak : { natural examples only have compositions (of 1-mor) associative only up to $\underbrace{\text{nat. isom}}_{\text{invertible 2-mor}}$.

<u>Theme</u> : $(\infty,0)$-cat $=$ homotopy types
$(\infty,1)$-cat $=$ Category theory $\cup$ homotopy theory
$=$ ho. type enriched cat

"category level"
$0 \le k \le \underbrace{n}$
"homotopy level"

We focus on $k=1$:
① This makes things simpler (behaves like "1-category")
② For many purposes $\underline{\text{invertible mor}}$ is what we care
↳ ho. thry.
③ Even if we are eventually interested in higher k, the theory of (∞,k)-cat is most efficiently developed in $(\infty,1)$-categorical language

<u>objects</u> $x, y, \dots$
<u>mor</u> $f : x \to y, \dots$
<u>2-mor</u>

$$\begin{tikzcd}
x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\cong"] & y
\end{tikzcd} , \; - - -$$

"homotopy

⋮

ex $X \in \mathrm{Top} \rightsquigarrow \Pi_{\le n} X$: fundamental n-groupoid ($(n,0)$-cat)
- obj : a point $x \in X$
- mor : a path $x \longrightarrow y$
- 2-mor : a homotopy between paths $x \overset{\Downarrow}{\rightrightarrows} y$
- 3-mor : a homotopy between homotopies
- ⋮
- n-mor : (ho. between $(n-1)$-mor.) / homotopy.

invertible

also $\Pi_{\le\infty} X$

$$\begin{tikzcd}
 & \mathrm{Gpd}_\infty \arrow[d] \\
 & \vdots \arrow[d] \\
 & \mathrm{Gpd}_2 = 2\text{-Type} \arrow[d] \\
 & \mathrm{Gpd} = 1\text{-Type} \arrow[d] \\
\mathrm{Top} \arrow[r, "\Pi_{\le 0}"'] \arrow[ru, "\Pi_{\le 1}"] \arrow[ruu, "\Pi_{\le 2}"] \arrow[ruuuu, "\Pi_{\le\infty}"] & \mathrm{Set} = 0\text{-Type}
\end{tikzcd}$$

ho. hyp
"<u>Def</u>" An ∞-groupoid $=$ a homotopy type

<u>Slogan</u> : homotopy theory $=$ the art of identification

<u>Terminology</u> ∞-groupoid $=$ anima $=$ (weak) homotopy types

all equivalent but sounds like / reflects the feeling of:
- special ∞-cat
- generalized groupoid
- some primitive notion like sets
- comes from a space

($=$ Kan complexes
$=$ spaces)

give representatives but should be avoided unless you really mean a particular one ex "the space / ∞-gpd of something" is connected $\Leftrightarrow$ unique up to iso

1-connected $\Leftrightarrow$ unique up to iso, unique up to homotopy

2-connected $\Leftrightarrow$ ______ " ______ , which is unique up to ho.

i.e. highly connected

$\Leftrightarrow$ a choice of such object is highly canonical

uniqueness in higher category theory = contractible space of choices

ex a choice of algebraic closure is unique up to iso but this iso is not unique.

the ∞-groupoid of choices $= BGal(\bar{k}/k)$ (profinite) 1-type.

Tricky part : ∞-gpd = anima should themselves form an ∞-category Ani

1-categorical approximation :

Def $\mathcal{H} := ho(Top) \simeq \begin{cases} \text{obj} & \text{CW cplx} \\ \text{mor} & \text{homotopy class of conti maps.} \end{cases}$

$\uparrow$ 1-category

The ∞-category Ani := Gpd$_\infty$ should fit into $Top \xrightarrow{\Pi_{\leq\infty}} Ani \xrightarrow{\pi_0 \text{ on Hom}} \mathcal{H} \xrightarrow{\pi_0} Set$ (with ho : $Top \to \mathcal{H}$ drawn above) and remembers

- all homotopical info of Top e.g.
  - for any $X, Y \in Ani$, the homotopy type $Map(X,Y) \in \mathcal{H}$, but also $\forall f, g : X \rightrightarrows Y$, ho. type $Map_{Map(X,Y)}(f,g)$, and so on.
  - Can test if a cone diagram is a homotopy (co) limit diagram
- but no more (homotopic maps should be "indisguinshable")

$Top \to \mathcal{H}$ is the initial functor to a 1-cat inverting w.ho.eq. (forgets too much for this)

expect : $Top \to Ani$ ―― " ―― ∞-cat ―― " ―― (remembers just enough)

these functors should be product-preserving (justified later ; homotopy product = product as the diagram shape is a set)

well-defined (2,1)-cat of enriched categories (up to categorical equiv.)

$$\begin{tikzcd}
\rightsquigarrow & Top\text{-}\widehat{Cat} \arrow[r] & \widehat{Cat}_\infty \arrow[r] \arrow[d, phantom, "="] & \mathcal{H}\text{-}\widehat{Cat} \arrow[r] & \widehat{Cat}_1 \arrow[ld, phantom, "\text{(large ver)}"] \\
 & & Ani\text{-}Cat & & \\
\text{feed itself :} & Top \arrow[u, phantom, "\in"] \arrow[r, mapsto] & Ani \arrow[u, phantom, "\in"] \arrow[r, mapsto] & \underline{\mathcal{H}} \text{ (self-enriched)} \arrow[uu, phantom, "\in"] \arrow[r, mapsto] & \mathcal{H} \arrow[uu, phantom, "\in"]
\end{tikzcd}$$

exer: check $Top \to \mathcal{H}$: product preserving & $\mathcal{H}$ is cartesian closed

⚠ otherwise $\mathcal{H}$ is an ill-behaved category e.g. has almost no (co) limits

Fact Top-Cat is strictly enriched, but it turns out that any ∞-category can be rectified & presented as a Top-Cat. So one might as well define ∞-cat as a Top-enriched cat

expect : $Top\text{-}Cat \to Cat_\infty$ is the initial functor to a ∞-cat which inverts DK-equivalences (ess. surj & hom-wise w.h.e.)

Rem

$\mathcal{H}$ is one step better approx for Ani. Heuristically, Ani is the limit of the iteration of this process. (the next step is $\mathcal{H}$-Gpd-enrichment) but such bottom-up approach suffers circular problem. (Notice the difficulty in $\mathcal{H}$ $\underbrace{\text{vs}}$ $\mathcal{H}$-Gpd)

However, a lot of ∞-categorical definition (limits, adjoints, ..) only depends on the underlying $\mathcal{H}$-enriched category.

Models of ∞-gpds & ∞-cats

To cut the circular chain. we need a 1-categorical defn.

$$\begin{tikzcd}
\text{1-categorical presentation} \arrow[r, squiggly] & \text{actual } \infty\text{-cat} \arrow[r, squiggly] & \text{1-categorical approx} \\
\text{(model categories / relative cats)} & \text{we're after} &
\end{tikzcd}$$

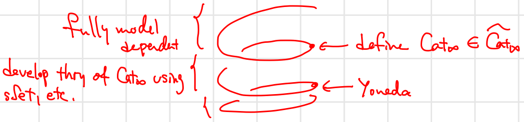
$$\begin{tikzcd}
(\overset{\mathrm{Top}}{\mathrm{CW}}, \overset{\mathrm{w.e.}}{\mathrm{ho.eq}}) \arrow[r] \arrow[rr, bend left] \arrow[d, shift right, "\mathrm{Sing}"'] & (\mathrm{Top}\text{-}\mathrm{Gpd}, \mathrm{DK}\text{-}\mathrm{eq}) \arrow[r] & \mathrm{Ani} = \mathrm{Gpd}_\infty \ (= \mathcal{S}) \arrow[r] & \mathcal{H} \\
(\underset{\mathrm{sSet.}}{\mathrm{Kan\ cpx}}, \underset{\text{Kan-Quillen eq.}}{\mathrm{ho.eq}}) \arrow[u, shift right, "|-|"'] \arrow[urr] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathrm{Top}\text{-}\mathrm{Cat}, \mathrm{DK\ eq}) \arrow[d, leftrightarrow, "\simeq"'] \arrow[ddr] & & \\
(\mathrm{Kan}\text{-}\mathrm{Cat}, \mathrm{DK\ eq}) \arrow[d, leftrightarrow, dashed] \arrow[dr] & & \\
\boxed{(\mathrm{qCat}, \mathrm{Joyal\ eq})} \arrow[r] & \mathrm{Cat}_\infty \arrow[r] & \mathcal{H}\text{-}\mathrm{Cat} \\
(\mathrm{RelCat}, \mathrm{w.e.}) \arrow[u, dashed] \arrow[ur] & \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq} & \\
\vdots & &
\end{tikzcd}$$

our choice

- choose a convenient / flexible def as the official def.
- undergo a few self-feeding spirals
- once you can fluently talk about Cat_∞ itself, you can forget the choice we made.

this work has been done so you can pretend you are here



§2 An implementation by simplicial sets

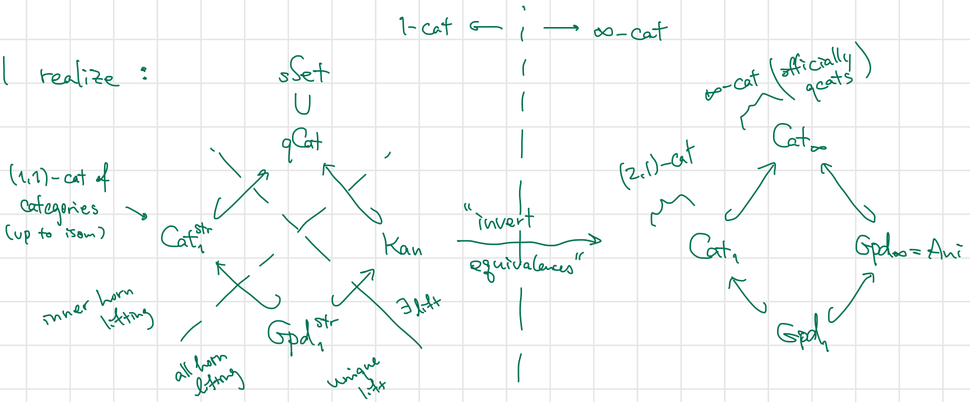
Def • $\Delta \subset \mathrm{Cat}$ finite nonempty ordinals. "Simplex category"
$\overset{\shortparallel}{\{[n] = \{0 < \cdots < n\} \mid n \geq 0\}}$

• A simplicial set is $X_\bullet : \Delta^{op} \longrightarrow \mathrm{Set}$. Let $\mathrm{sSet} := \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$.

$x \in X_n$: an n-simplex of X. $\longleftrightarrow \underset{\underset{y[n]}{\shortparallel}}{\Delta^n} \longrightarrow X$

$(\rightsquigarrow \{k\text{-simplex of } \Delta^n\} = \{[k] \to [n]\}$ $\left(= \text{possibly degenerate } k\text{-simplex of a simplicial complex "}\Delta^n\text{"}\right)$

We will realize :



Recall $\mathcal{D}$: cocomplete

$$\begin{tikzcd} \Delta \arrow[r, "f"] \arrow[d, "y"'] & \mathcal{D} \arrow[dl, bend left, "f^*"] \\ \mathrm{Fun}(\Delta^{op}, \mathrm{Set}) \arrow[ur, bend left, "f_!"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

• $f_!$: unique colim-pres extension of f
$\left(f_! X = \underset{\Delta^n \to X}{\mathrm{colim}}\, f[n]\right)$

• f^* : right adj given by $f^*d : \Delta^{op} \longrightarrow \mathrm{Set}$
$[n] \longmapsto \mathrm{Hom}_{\mathcal{D}}(f[n], d)$

ex.1

$$\begin{tikzcd} \Delta \arrow[r, hook] \arrow[d] & \mathrm{Cat}_1^{str} \arrow[dl, bend left, "N"] \\ \mathrm{sSet} \arrow[ur, bend left, "ho"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

• $(N\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C}) \in \mathrm{Set}$

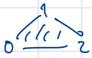
exer N is fully faithful $\left(\Leftrightarrow \Delta \text{ is } \overset{\text{(1-categorically)}}{\text{dense}} \text{ in } \mathrm{Cat}_1^{str}\right)$
i.e. $ho \circ N(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$
(↳ we will omit N and regard strict 1-cat as a simplicial set)

i^{th} horn

Def • $\Lambda^n_i \subset \Delta^n$ sub sset of those $[k] \to [n]$ s.t. Image $\not\supset [n] \setminus \{i\}$
(i.e. the top cell & the i^{th} face removed)

$i = n$ right
$0 < i < n$ inner
$i = 0$ left
} horn

(ex : $\Lambda^2_1 =$


$\subset$


$= \Delta^2$)

(Inner anodyne is a closure of inner horns under pushouts, trfin composition, retracts

- $\mathrm{Spine}_n = \Delta^1 \vee \Delta^1 \vee \cdots \vee \Delta^1 \hookrightarrow \Delta^n$

exer (a) TFAE :

(1) $\exists \mathscr{C} \quad X \simeq N\mathscr{C}$

(2) $0 < \forall i < \forall n$

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

i.e.

$$\begin{tikzcd} \mathrm{Hom}(\Delta^n, X) \arrow[d, "\cong\ \mathrm{bij}"] \\ \mathrm{Hom}(\Lambda^n_i, X) \end{tikzcd}$$

(3) $\forall n \quad \mathrm{Hom}(\Delta^n, X) \xrightarrow{\cong} \mathrm{Hom}(\mathrm{Spine}_n, X)$

(b) TFAE :

(1) $\exists \mathscr{C}$: groupoid $X \simeq N\mathscr{C}$

(2) $\forall n \geq 1,\ 0 \leq \forall i \leq n \quad \mathrm{Hom}(\Delta^n, X) \xrightarrow{\mathrm{bij}} \mathrm{Hom}(\Lambda^n_i, X)$

Def define full sub $\mathrm{Kan} \subset q\mathrm{Cat} \subset s\mathrm{Set}$ by

$X \in q\mathrm{Cat}$ (Kan) $\iff 0 < \forall k < \forall n$ $\begin{cases} 0 \leq \forall k \leq \forall n. \\ n \geq 1 \end{cases}$

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

i.e.

$$\begin{tikzcd} \mathrm{Hom}(\Delta^n, X) \arrow[d, "\mathrm{surj.}"] \\ \mathrm{Hom}(\Lambda^n_k, X) \end{tikzcd}$$

ex. 2

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d] & \mathrm{Top} \arrow[dl, bend left, "\mathrm{Sing}"] \\ s\mathrm{Set} \arrow[ur, "1\text{-}1"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

exer $\mathrm{Sing}(X)$ is a Kan complex. (observe the non-uniqueness of ext)

fact · " $f_0, f_1 : X \rightrightarrows Y$ homotopic $\iff \exists$

$$\begin{tikzcd} \partial\Delta^1 \times X \arrow[d] \arrow[dr, "f_0 \sqcup f_1"] & \\ \Delta^1 \times X \arrow[r] & Y \end{tikzcd}$$

"

defines congruence on Kan.

· this restricts to $\mathrm{Kan} \underset{\perp}{\rightleftarrows} \mathrm{CW}$ respecting ho.eq.

($\Leftarrow$ 1-1 product preserving)

$\rightsquigarrow \mathrm{ho}(\mathrm{Kan}) \simeq \mathrm{ho}(\mathrm{CW}) \simeq \mathcal{H}$ (gives a combinatorial model of homotopy types)

Moreover, this lifts to a Q.E. of model cats

$(s\mathrm{Set}, \text{Kan-Quillen}) \rightleftarrows (\mathrm{Top}, \text{Serre fib. etc.})$

So the LHS knows all the (weak) homotopy theory of top. sp.

Def An ∞-groupoid / anima is a Kan complex.

An ∞-category is a quasi-category

A functor = mor of ssets

⚠ not talking about ∞-cats of those yet.

justification

$\mathrm{Spine}_2 = \Lambda^2_1 =$

$$\begin{tikzcd} & 1 \arrow[dr] & & & & y \arrow[dr, "g"] & \\ 0 \arrow[ur] & & 2 \arrow[rrr] & & x \arrow[ur, "f"] & & z \end{tikzcd} \quad X$$

$$\begin{tikzcd} \Lambda^2_1 \arrow[d, hook] \arrow[r] & X \\ \Delta^2 \arrow[ur, dashed, "\exists"] & \end{tikzcd}$$

$\Delta^2 =$

$$\begin{tikzcd} & 1 \arrow[dr] & & & & y \arrow[dr, "g"] & \\ 0 \arrow[ur] \arrow[rr] \arrow[rr, phantom, bend left=20, "(1)"] & & 2 & & x \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, bend left=20, "\alpha"] & & z \end{tikzcd}$$

part of data

think: α "witness" the composition $h \simeq g \circ f$.

ex in fundamental $(\infty\text{-})$ groupoid, any reparametrization of path concatenation is equally good as a composition.

higher horn filling condition $\iff$ the choice of compositions is contractible.

Fact sSet has the internal hom $[K, X]$ (given by $[K, X]_n = [\Delta^n \times K, X]$)
if X is a Kan cpx or a qcat, so is $[K, X]$.

Def For $X \in \mathrm{qCat}$, $\mathrm{Fun}(K, X) := [K, X]$ "functor / diagram category"

Def $\mathrm{Kan} \hookrightarrow \mathrm{qCat}$ admits a right adjoint $X \mapsto X^{\simeq}$ maximal sub Kan complex
$\mathrm{Map}(K, X) := \mathrm{Fun}(K, X)^{\simeq}$

Fact $\exists$ qcat / Kan cplx completion
$\mathrm{sSet} \longrightarrow \mathrm{qCat} \longrightarrow \mathrm{Kan}$
(fibrant replacement via inner anodyne.)

Fact TFAE: (1) $X \in \mathrm{qCat}$

+ univalence ?

(2)
$$\begin{tikzcd}[\Delta^2, X] \arrow[d] \\ {[\Lambda^2_1, X]}\end{tikzcd}$$
is a trivial fibration (i.e. bundle of contractible Kan cplx)

(3) $\forall n$
$$\begin{tikzcd}[\Delta^n, X] \arrow[d, shift right, "\simeq"'] \\ {[\mathrm{Spine}_n, X]} \arrow[u, shift right]\end{tikzcd}$$
is a trivial Kan fib.

we may choose a section so we have "the" composition

$$\begin{tikzcd}\bullet \arrow[r, "f_1"'] \arrow[rrrrr, bend right, "f_n \circ \cdots \circ f_1"'] & \bullet \arrow[r, "f_2"'] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r, "f_n"'] & \bullet\end{tikzcd}$$
$\rightsquigarrow$ for $(\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet \to \cdots \bullet \xrightarrow{f_n} \bullet)$, $f_n \circ \cdots \circ f_1$.

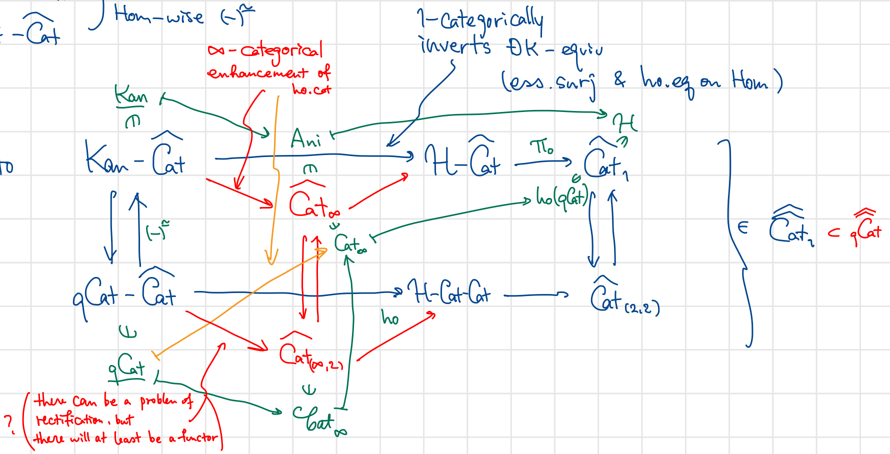
Joyal eq. $:\Longleftrightarrow$ isom in $\mathrm{ho}(\mathrm{qCat}) := \begin{cases} \mathrm{obj} : \mathrm{qcat} \\ \mathrm{mor} : \pi_0 \mathrm{Fun}(X, Y)^{\simeq} \end{cases}$

$\Longleftrightarrow A \to B$ s.t. $\mathrm{Map}(B, \mathcal{C}) \xrightarrow{\sim} \mathrm{Map}(A, \mathcal{C})$

∞-Cat of ∞-cats ?

$\underline{\mathrm{Kan}} \overset{\mathrm{Map}}{\in} \widehat{\mathrm{Kan\text{-}Cat}}$
$\underline{\mathrm{qCat}} \underset{\mathrm{Fun}}{\in} \widehat{\mathrm{qCat\text{-}Cat}}$ $\Big\}$ Hom-wise $(-)^{\simeq}$

These fit into



We can do this now!

Def A map of qcats $f : \mathcal{C} \to \mathcal{D}$ is a $\underline{\text{Joyal eq}}$ if $\simeq$ in $\mathrm{ho}(\mathrm{qCat})$
A map of Kan-cats — " — $\underline{\text{DK-eq}}$ if $\simeq$ in H-Cat

Our task
- ∞-categorical localization $\rightsquigarrow \mathrm{Cat}_\infty = \mathrm{qCat}[\text{Joyal eq}^{-1}]$
- Compare Kan-Cat to qCat to close the self-feeding loop $\rightsquigarrow$

$$\begin{tikzcd}\mathrm{qCat} \arrow[dr, "\text{Joyal eq}^{-1}"] & \\ & \mathrm{Cat}_\infty \\ \text{Kan-Cat} \arrow[uu, "N_{hc}"] \arrow[ur, "\text{DK-eq}^{-1}"'] & \end{tikzcd}$$

§ Relative categories & localizations

Def A relative $(\infty\text{-})$category is a pair $(\mathcal{C}, W)$ where $\mathcal{C}$: $(\infty\text{-})$Category / poset

W : a collection of morphisms of $\mathcal{C}$.

poset

Def $\mathrm{Fun}((\mathcal{C},S),(\mathcal{D},T)) \subset \mathrm{Fun}(\mathcal{C},\mathcal{D})$ full sub spanned by $\mathcal{C} \xrightarrow{f} \mathcal{D}$ s.t. $f(S) \subset T$.

└ in qcat: sub sset with simplices only containing the prescribed vertices

Def A functor $\mathcal{C} \xrightarrow{f} \mathcal{D}$ exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ wrt W (or

$$\begin{tikzcd} \Longleftrightarrow \ \forall \mathcal{E} \in \mathrm{Cat}_\infty \quad \mathrm{Fun}(\mathcal{D},\mathcal{E}) \arrow[r, "f^*"'] \arrow[dr, "\circledast", "\simeq"'] & \mathrm{Fun}(\mathcal{C},\mathcal{E}) \text{ is fully faithful w/ ess. image} \\ & \mathrm{Fun}((\mathcal{C},W),(\mathcal{E},\mathcal{E}^{\simeq})) =: \mathrm{Fun}^W(\mathcal{C},\mathcal{E}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

— only for now

- $\mathrm{ho}\,\mathcal{C}[W^{-1}]$ is the 1-categorical localization.

Rem
- $\forall \mathcal{E}$ f^* factors through $\mathrm{Fun}^W(\mathcal{C},\mathcal{E}) \Leftrightarrow f(W) \subset \mathcal{D}^{\simeq}$
- under the condition $f(W) \subset \mathcal{D}^{\simeq}$, it is enough to ask $\circledast$ to be an equivalence of underlying anima $(\mathrm{Map} = \mathrm{Fun}^{\simeq})$ or even $\pi_0\mathrm{Map}$.

☺ $\mathrm{Fun}(\mathcal{D},\mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^W(\mathcal{C},\mathcal{E}) \Longleftarrow \forall \mathcal{B} \in \mathrm{Cat}_\infty$

$$\begin{tikzcd} \pi_0\mathrm{Fun}(\mathcal{B},\mathrm{Fun}(\mathcal{D},\mathcal{E})) \arrow[r, "\simeq"] \arrow[d, phantom, "\shortparallel"] & \pi_0\mathrm{Fun}(\mathcal{B},\mathrm{Fun}^W(\mathcal{D},\mathcal{E})) \arrow[d, phantom, "\shortparallel"] \\ \pi_0\mathrm{Fun}(\mathcal{D},\underbrace{\mathrm{Fun}(\mathcal{B},\mathcal{E})}_{\mathcal{E}'}) \arrow[r] & \pi_0\mathrm{Fun}^W(\mathcal{D},\underline{\mathrm{Fun}(\mathcal{B},\mathcal{E})}) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{RelCat}_\infty \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_\infty \arrow[l, shift left] \\ (\mathcal{E},\mathcal{E}^{\simeq}) \arrow[u, phantom, "\cup"] & \mathcal{E} \arrow[u, phantom, "\cup"] \arrow[l, mapsto] \end{tikzcd}$$

localization $\mathcal{C}[W^{-1}]$
= local adjoint at $(\mathcal{C},W)$
(unique up to contractible choice)

Prop. $\forall (\mathcal{C},W) \in \mathrm{RelCat}_\infty$. the localization $\mathcal{C} \to \mathcal{C}[W^{-1}]$ exist.

Proof

$$\begin{tikzcd} \coprod_W \Delta^1 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \coprod_W * \arrow[r] & \mathcal{C}[W^{-1}] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in Cat_∞

(note: $\Delta^1 \to *$ is epi corepresenting)

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Delta"] \arrow[d, phantom, "\wr"] & \mathcal{C}^{\Delta^1} \\ \mathrm{Isom}(\mathcal{C}) \arrow[ur, hook] & \end{tikzcd}$$

also: $\mathcal{C} \longmapsto (\mathcal{C},\mathcal{C})$

$$\begin{tikzcd} \mathrm{Cat}_\infty \arrow[r] & \mathrm{RelCat} \arrow[d] \\ \mathrm{Ani} \arrow[u, "\text{1-1}"'] \arrow[r, hook] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Cat}_\infty \end{tikzcd}$$

exer

$$\begin{tikzcd} \Delta^{\{0<2\}} \sqcup \Delta^{\{1<3\}} \arrow[r] \arrow[d] & \Delta^3 \arrow[d] \\ \Delta^0 \sqcup \Delta^0 \arrow[r] & \mathrm{Isom} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[d, "\simeq"] \\ & \Delta^0 \end{tikzcd}$$

— equiv.

Fact $\mathrm{RelCat} \longrightarrow \mathrm{Cat}_\infty$ is essentially surjective.

- ∃ model str. w/ cofibrant = RelPoSet (check Barwick-Kan)

w.e. = DK-eq after Hammock loc.

$$\begin{tikzcd} \mathrm{RelCat} \arrow[r] \arrow[d, "\text{Hammock localization}"'] & \mathrm{Cat}_\infty \\ \text{sSet-Cat} \arrow[r, "N_\Delta"] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{sSet} \arrow[u] \end{tikzcd}$$

cf.

$$\begin{tikzcd} \mathrm{PoSet} \arrow[r, "\text{1-1}"] \arrow[d, "N"] & \mathrm{Ani} \\ \mathrm{sSet} \arrow[ur, "\text{1-1}"'] & \end{tikzcd}$$

ess. surj.

$\mathrm{sSet}[\mathrm{w.h.e.}^{-1}] \simeq \mathrm{Ani}$.

$\left(\mathrm{RelPos}[\mathrm{w.e.}^{-1}] \simeq\right) \mathrm{Kan}\text{-}\mathrm{Cat}[\mathrm{DK\ eq}^{-1}] \underset{\uparrow \text{next}}{\simeq} \mathrm{qCat}[\mathrm{Joyal\ eq}^{-1}] \simeq \mathrm{Cat}_\infty$

§ Simplicially enriched cats as ∞-categories

ex.3

$$\begin{tikzcd} {[n]} \arrow[r, mapsto] & \mathfrak{C}[n] \\ \Delta \arrow[r] \arrow[d] & \text{sSet-Cat} \arrow[dl, bend left, "N_{hc}"] \\ \text{sSet} \arrow[ur, bend left, "\mathfrak{C}"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

N_{hc}: homotopy coherent nerve

"homotopy coherent realization of $[n]$"

$\mathfrak{C}[n]$: obj $\{0, \dots, n\}$

$\mathrm{Hom}_{\mathfrak{C}[n]}(i, j) = \left\{ \begin{array}{l} \text{a path from} \\ i \text{ to } j \text{ in } [n] \end{array} \right\}$

poset by refinement. $(\simeq [1]^{\#\{k \mid i<k<j\}})$

↳ $\text{Cat-Cat} \xrightarrow{N} \text{sSet-Cat}$

Facts · Joyal eq ⟷ DK eq under $\mathfrak{C} \dashv N_{hc}$ (at least between qcat / Kan-Cat) In fact, these are part of Quillen eq. of model cats ⇝ enough for ≃ of ∞-categorical localizations

· $$\begin{tikzcd} \text{sSet-Cat} \arrow[r, "N_\Delta"] & \text{sSet} \\ \text{Kan-Cat} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & \text{qCat} \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

$\mathrm{ho}(\text{Kan-Cat}, \text{DK-eq}) \simeq \mathrm{ho}(\text{qCat}, \text{Joyal eq})$

(use: $\mathfrak{C}$[inner horn incl.] = hom-wise $\sqcap \hookrightarrow \square$ ← exer check when $n=3$

Def $\mathrm{Cat} := \mathrm{Cat}_\infty := N_\Delta(\text{qCat})$.

$\quad\cup$

$\mathrm{Ani} := N_\Delta(\mathrm{Kan})$

Def $X \in \text{qCat}$, $x_0, x_1 \in X$ ⇒

$$\begin{tikzcd} \mathrm{Map}_X(x_0, x_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, X) \arrow[d] \\ * \arrow[r, "{(x_0, x_1)}"'] & X \times X = \mathrm{Fun}(\partial\Delta^1, X) \end{tikzcd}$$

qCat ↓ $\mathcal{H}$-Cat.

Kan ↗ $\mathrm{Map}_X(x_0, x_1)$

← **exer**: this is a Kan fib. (RLP wrt ∀ horn incl.)

Fact $\mathcal{C} \in \text{Kan-Cat}$, $x, y \in \mathcal{C}$ ⇝ $\mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{\simeq} \mathrm{Map}_{N_\Delta \mathcal{C}}(x, y)$

(kerodon 01LA)

$\underset{\mathcal{C} = \mathrm{Cat}_\infty}{\rightsquigarrow}$ $\boxed{\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{A}, \mathcal{B}) \xleftarrow{\simeq} \mathrm{Fun}(\mathcal{A}, \mathcal{B})^{\simeq}}$

Compute this by simp. cptd tech

↓

$\mathfrak{C}\left[\dfrac{\Delta^1 \times \Delta^n}{\partial\Delta^1 \times \Delta^n}\right]$ vs $[1](\Delta^n) \in \text{sSet-Cat}$

Rem It is possible to formulate $(\infty,2)$-categories in sSet &

$\text{qCat-Cat} \xrightarrow[N_\Delta]{} (\infty,2)\mathrm{Cat}$

pretend to, though not quite yet (until yoneda)

From now on: We work model-agnostically, internally to Cat

Slogan: as long as universal constructions go, ∞-cats are like 1-cats.

(we can still use functors from qCat, Kan-Cat to produce examples

exceptions: Small combinatorial computation (e.g. to establish a special property of Ani / qCat when difficult to avoid, isolate as a formula)

Simplicial sets as diagram shapes (⟷ graph in 1-cats)

∀ sset: $$\begin{tikzcd} K \arrow[r] \arrow[dr, "\sim"] & \mathcal{C} \\ & \mathcal{D} \arrow[u] \end{tikzcd}$$

inner anodyne

↑

Joyal eq

"Category completion" e.g. $\mathrm{Spine}[n] \to \Delta^n$

§ Underlying $\mathcal{H}$-cats

Takeaway:

$$\begin{tikzcd}
\mathrm{Kan}\text{-}\mathrm{Cat} \arrow[r] \arrow[d] & \mathrm{Cat} \arrow[r, "\underline{ho}"] \arrow[dd, red, bend right, "{|-|}"'] \arrow[dd, bend left, "(-)^{\simeq}"] & \mathcal{H}\text{-}\mathrm{Cat} \arrow[r] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{Cat}_1 \arrow[ll, bend left, "N"] \arrow[dd, bend right, "\dashv"'] \arrow[dd, bend left, "\dashv"] \\
\mathrm{qCat} \arrow[ur] \arrow[d, red, dashed, bend right] \arrow[d, shift left, "\dashv"] & & & \\
\mathrm{Kan} \arrow[r] \arrow[u, hook, shift left] & \mathrm{Ani} \arrow[uu, hook, "\dashv"'] \arrow[rr, "ho = \pi_{\leq 1}"] \arrow[rr, phantom, shift right=3, "\perp"] & & \mathrm{Gpd}_1 \arrow[uu, hook] \arrow[ll, bend left, "N"]
\end{tikzcd}$$

We'll see:

$\mathrm{qCat} \longrightarrow \mathrm{Cat} \longrightarrow \mathcal{H}\text{-}\mathrm{Cat}$

$\mathrm{Kan} \longrightarrow \mathrm{Ani} \longrightarrow \mathcal{H}$

product preserving

- $$\begin{tikzcd}
\mathscr{C}^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{C} \arrow[d] \\
ho(\mathscr{C})^{\simeq} \arrow[r] & ho(\mathscr{C})
\end{tikzcd}$$ ← Conservative

- The internal hom $\mathrm{Fun}(\mathscr{C}, \mathrm{Fun}(\mathscr{D}, \mathscr{E})) \simeq \mathrm{Fun}(\mathscr{C}\times\mathscr{D}, \mathscr{E})$

  ($\rightsquigarrow \mathrm{Cat}_\infty$: $(\infty, 2)$-cat)

  $h_2\mathrm{Cat}$: homotopy 2-cat ($\simeq h_2\mathrm{QCat}$)

- $\mathscr{C} \ni x, y \rightsquigarrow$
  $$\begin{tikzcd}
  \mathrm{Map}_{\mathscr{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\Delta^1, \mathscr{C}) \arrow[d] \\
  * \arrow[r, "{(x,y)}"'] & \mathrm{Fun}(\partial\Delta^1, \mathscr{C})
  \end{tikzcd}$$
  $\mathrm{Ani} \ni$

  later: $\mathrm{Map}_{\mathscr{C}}(-,-) : \mathscr{C}^{op} \times \mathscr{C} \longrightarrow \mathrm{Ani}$

e.g. $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathscr{C}, \mathscr{D}) \simeq \mathrm{Fun}(\mathscr{C}, \mathscr{D})^{\simeq}$ by construction

Many notion can be detected in $\underline{ho}(\mathscr{C})$ once data are provided in $\mathscr{C}$ ($\in \mathrm{Cat}$)

$\underline{\text{ex}}$ $f : X \to Y$ isom in $\mathscr{C}$ $:\Longleftrightarrow$ $[f] : [X] \xrightarrow{\cong} [Y]$ in $ho(\mathscr{C})$ in fact,
$$\begin{tikzcd}
\mathscr{C}^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{C} \arrow[d] \\
ho(\mathscr{C})^{\simeq} \arrow[r] & ho(\mathscr{C})
\end{tikzcd}$$

$\underline{\text{ex}}$ $\mathscr{C} \xrightarrow{f} \mathscr{D} \in \mathrm{Cat}$: fully faithful $:\Longleftrightarrow$ $\underline{ho}(f)$: ff

ess surj $:\Longleftrightarrow$ $\underline{ho}(f)$: e.s. $\Longleftrightarrow$ $ho(f)$: e.s.

isom in Cat $\Longleftrightarrow$ $\underline{ho}(f)$: eq

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "f"] \arrow[dr, red, "es"'] & \mathscr{D} \\
& \text{ess.im}(f) \arrow[u, red, "f.f"']
\end{tikzcd}$$

∃! factorization

↳ tautological for Kan-Cat. I don't see a model-indep proof, but at least:

$\underline{\text{exer}}$ $\underline{ho}(f)$: eq $\Longleftrightarrow$ $\mathrm{Map}(\Delta^n, f)$: isom in Ani for $n = 0, 1$ (actually $n=0$ follow from $n=1$ as Δ^0 is a retract of Δ^1)

($\Leftarrow$ Δ^1 is a colimit-generator of Cat)

$\underline{\text{exer}}$ • $F : \mathscr{C} \longrightarrow \mathscr{D}$ $\in$ Ani is (1) fully faithful iff it is an incl. of conn. components

(2) ess.surj. iff it is π_0-surjection.

(3) equiv iff ho. eq.

§ (co)limits / adjunction

If $K \in \mathrm{sSet}$ (or Cat_∞), $K \to *$ induces the diagonal functor $\delta : \mathcal{C} \xrightarrow{x \longmapsto \underline{x}} \mathrm{Fun}(K, \mathcal{C})$.

Def Let $K \xrightarrow{f} \mathcal{C}$ be a diagram. A natural transformation $\underline{x} \xrightarrow{\varepsilon} f$ exhibits $x \xrightarrow{\sim} \lim f$ if the composition

$$\mathrm{Map}_{\mathcal{C}}(y, x) \xrightarrow[\delta]{} \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{y}, \underline{x}) \xrightarrow[\varepsilon]{} \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{y}, f)$$

is an isom in Ani. ($\Leftrightarrow$ in $\mathcal{H}$).

Colimits are defined similarly.

ex $K = \phi \rightsquigarrow$ terminal / initial obj : $\mathrm{Map}(y, *) = *$, $\mathrm{Map}(\phi, y) = *$

$\lim_{\Delta}$ totalization, $\mathrm{colim}_{\Delta^{op}}$: geom. realization

ex $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ computed pointwise.

Exer Ani $\longrightarrow \mathcal{H}$ preserves products & coproducts

(co)limits are special cases of local adjoints:

Def • Let $F : \mathcal{C} \to \mathcal{D}$. A right adjoint of F at $x \in \mathcal{D}$ is a pair $(\overset{\text{“}F^R x\text{”}}{y} \in \mathcal{C}, \underset{\in \mathcal{D}}{Fy \xrightarrow{\varepsilon} x})$ s.t. $\forall z \in \mathcal{C}$, the composition

$$\mathrm{Map}_{\mathcal{C}}(z, y) \xrightarrow[F]{} \mathrm{Map}_{\mathcal{D}}(Fz, Fy) \xrightarrow[\varepsilon_*]{} \mathrm{Map}_{\mathcal{D}}(Fz, x)$$ is an iso in Ani.

• A (global) right adjoint of F is a pair $(F^R : \mathcal{D} \to \mathcal{C}, FF^R \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{D}} \in \mathrm{Fun}(\mathcal{D}, \mathcal{D}))$ s.t. $\forall c \in \mathcal{C}$, the composition

$$\mathrm{Map}_{\mathcal{C}}(c, \underset{Fc'}{F^R d}) \longrightarrow \mathrm{Map}_{\mathcal{D}}(Fc, FF^R d) \xrightarrow[\varepsilon_*]{} \mathrm{Map}_{\mathcal{D}}(Fc, d)$$ is an iso.] ⊛

Fact $\exists$ local adj at every $d \in \mathcal{D} \rightsquigarrow$ can assemble into a global adj.

Rem ⊛ can be replaced by $\exists \eta : \mathrm{id}_{\mathcal{C}} \longrightarrow F^R F$ s.t.

$$\begin{tikzcd}[column sep=small] & FF^RF \arrow[dr, "\varepsilon F"] \arrow[d, phantom, "\simeq"] & \\ F \arrow[ur, "F\eta"] \arrow[rr, "="'] & {} & F \end{tikzcd} , \quad \begin{tikzcd}[column sep=small] & F^RFF^R \arrow[dr] \arrow[d, phantom, "\simeq"] & \\ F^R \arrow[ur] \arrow[rr, "="'] & {} & F^R \end{tikzcd}$$

adjoints are detected in $h_2\mathrm{Cat}$.

} use Yoneda.

Def $F : \mathcal{C} \to \mathcal{D}$ preserves limits $\Leftrightarrow \forall K \overset{G}{\underset{\underline{x}}{\rightrightarrows}} \mathcal{C} \to \mathcal{D}$ $\underline{x} \to G$ limit cone $\Rightarrow$ $\underline{Fx} \Rightarrow FG$ lim. cone

reflects $\Leftarrow$

exer left adj pres colim, right adj pres lim

Rem $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ ($\perp$) $\Rightarrow \mathrm{Fun}(K, \mathcal{C}) \underset{R\circ -}{\overset{L\circ -}{\rightleftarrows}} \mathrm{Fun}(K, \mathcal{D})$ ($\perp$)

§ join & slice (computation here are done in sSet)

Def In Cat, oriented pushout

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "c"] \\
\mathcal{B} \arrow[r, "b"'] \arrow[ur, Rightarrow, shorten=3mm, "\alpha"'] & \mathcal{B} \overrightarrow{\sqcup}_{\mathcal{A}} \mathcal{C}
\end{tikzcd}
\quad :\Longleftrightarrow \quad
\begin{tikzcd}
 & \mathcal{A} \arrow[r] \arrow[d, hook, "i_1"] & \mathcal{C} \arrow[dd, "c"] \\
\mathcal{A} \arrow[r, hook, "i_0"'] \arrow[d] & \mathcal{A} \times \Delta^1 \arrow[dr, "\alpha"] & \\
\mathcal{B} \arrow[rr, "b"'] & & \mathcal{B} \overrightarrow{\sqcup}_{\mathcal{A}} \mathcal{C}
\end{tikzcd}$$

colim diag

ori. pb

$$\begin{tikzcd}
\mathcal{B} \overrightarrow{\times}_{\mathcal{D}} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathcal{B} \arrow[r] \arrow[ur, Rightarrow, shorten=3mm] & \mathcal{D}
\end{tikzcd}
\quad :\Longleftrightarrow \quad
\begin{tikzcd}
\mathcal{B} \overrightarrow{\times}_{\mathcal{D}} \mathcal{C} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{C} \arrow[d] \\
 & \mathcal{D}^{\Delta^1} \arrow[r] \arrow[d] & \mathcal{D} \\
\mathcal{B} \arrow[r] & \mathcal{D} &
\end{tikzcd}$$

lim diag

fact

$$\rightsquigarrow \quad \mathrm{Fun}(\mathcal{B} \overrightarrow{\sqcup}_{\mathcal{A}} \mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{A}, \mathcal{B} \overrightarrow{\times}_{\mathcal{D}} \mathcal{C})$$

ex the join

$$\begin{tikzcd}
K \times L \arrow[r, "pr"] \arrow[d, "pr"'] & L \arrow[d] \\
K \arrow[r] \arrow[ur, Rightarrow, shorten=3mm] & K \star L
\end{tikzcd}$$

the slice of $K \xrightarrow{f} \mathcal{C}$:

$$\begin{tikzcd}
\overset{\mathcal{C}_{f/}}{\mathcal{C}_{/f}} \arrow[r] \arrow[d] & \{f\} \arrow[d] \\
\mathcal{C} \arrow[r] \arrow[ur, Rightarrow, shorten=3mm, shift left] \arrow[ur, Leftarrow, shorten=3mm, shift right] & \mathrm{Fun}(K, \mathcal{C})
\end{tikzcd}$$

$$K^{\triangleright} = K \star \underset{\substack{\uparrow \\ \text{terminal}}}{*}, \quad K^{\triangleleft} = \underset{\substack{\uparrow \\ \text{initial}}}{*} \star K$$

freely adjoin

$$* \xrightarrow{x} \mathcal{C} \rightsquigarrow \mathcal{C}_{/x}, \mathcal{C}_{x/}$$

fact associative up to cat eq.

fact $\mathrm{Fun}_{K/}(\underset{L \star K}{K \star L}, \mathcal{C}) \simeq \mathrm{Fun}(L, \underset{\mathcal{C}_{/f}}{\mathcal{C}_{f/}})$

Rmk $\lim f$ (resp. $\mathrm{colim}\, f$) is the final (resp. initia object of $\mathcal{C}_{/f}$ (resp. $\mathcal{C}_{f/}$)

fact $\mathcal{C}_{f/} \longrightarrow \mathcal{C}$ creates limits , $\mathcal{C}_{/f} \longrightarrow \mathcal{C}$ creates colim

(☺ reduces to $K \star L \sqcup_{L} L^{\triangleright} \longrightarrow K \star L^{\triangleright}$ is an inner anodyne. , I think)

exer f is a $\underset{\text{left}}{\text{right}}$ adjoint $\Longleftrightarrow \forall d \in \mathcal{D}$ $\underset{\mathcal{C}_{/d}}{\mathcal{C}_{d/}}$ has a $\underset{\text{terminal}}{\text{initial}}$ obj | i.e.

$$\begin{tikzcd}
c \arrow[r, mapsto] \arrow[d, phantom, "\cap"] & d \to fc \arrow[d, phantom, "\cap"] \\
\mathcal{C}_{c/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\dashv"] & \mathcal{D}_{d/} \arrow[d] \\
c \in \mathcal{C} \arrow[r, "f"'] & \mathcal{D}
\end{tikzcd}$$

exhibits c as $f^R d$

Formula

$$\begin{tikzcd}
 & \Lambda^n_n \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \Lambda^n_n \star K \arrow[d, hook] \\
\partial\Delta^n \star * \arrow[d, hook] & \Delta^n \arrow[r, hook] & \Delta^n \star K \\
\Delta^n \star * \arrow[ur, squiggly] & &
\end{tikzcd}$$

in Cat.

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & L^{\triangleright} \arrow[d, hook] \\
K \star L \arrow[r] & (K \star L)^{\triangleright}
\end{tikzcd}$$

what is the ultimate formula for these combinatorics?

Grothendieck construction

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\text{Some fibration}"'] \arrow[dr, phantom, "\lrcorner"] & \widetilde{\mathcal{U}} \arrow[d, "\text{tautological fib}"] \\ B \arrow[r] & \mathcal{U} \end{tikzcd}$$

"moduli of fibers"

$$\begin{tikzcd} Et(\mathcal{F}) \arrow[d] & \\ X \arrow[r, "\mathcal{F}"', "\text{sheaf}"'] & Set \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & EGL(n) \times_{GL(n)} \mathbb{R}^n \arrow[d] \\ X \arrow[r] & \{\mathbb{R}^n \in Vect\}^{\simeq} = BGL(n) \end{tikzcd} \qquad \begin{tikzcd} \int F = Y \arrow[r] \arrow[d, "Set \ni"'] \arrow[dr, phantom, "\lrcorner"] & Set_* \arrow[d] \\ X \arrow[r, "F"'] & Set \end{tikzcd}$$

Q. What do Ani & Cat classify?

functor $\Delta^n \xrightarrow{F} Cat = N_{hc}(qCat)$

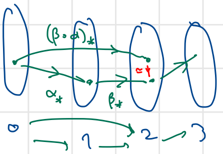
$\mathfrak{C}(\Delta^n) \longrightarrow qCat$ i.e. ho. ch. diag

$\underline{n=3}$

$$\begin{tikzcd} & F(1) \arrow[r] & F(2) \arrow[dr] & \\ F(0) \arrow[ur] \arrow[urr] \arrow[rrr] & \arrow[phantom, "\simeq"] & \arrow[phantom, "\simeq"] & F(3) \end{tikzcd} \simeq \begin{tikzcd} & F(1) \arrow[r] \arrow[drr] & F(2) \arrow[dr] & \\ F(0) \arrow[ur] \arrow[rrr] & \arrow[phantom, "\simeq"] & \arrow[phantom, "\simeq"] & F(3) \end{tikzcd}$$

This is hard to specify : any compatibly equivalent choice will be equally good, but all the choice must be made

$$\begin{tikzcd} \int F \arrow[d] \\ \Delta^n \end{tikzcd}$$



In fibration picture, $F(i) \xrightarrow{\alpha_*} F(j)$ are given by "transport" along $i \xrightarrow{\alpha} j$ described by a $\boxed{\text{univ. property}}$

also coherence is "automatic" ⟵ $\alpha_* \beta_* \simeq (\alpha\beta)_*$ etc.

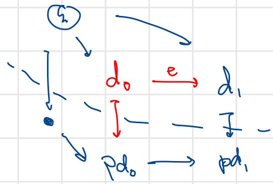
(⟿ e is uniquely determined as the initial obj of the pullback (if exists))

A functor

Def $$\begin{tikzcd} \mathcal{D} \arrow[d, "p"'] & d_0 \arrow[r, "e"] \arrow[d, mapsto] & d_1 \arrow[d, mapsto] \\ \mathcal{C} & pd_0 \arrow[r] & pd_1 \end{tikzcd}$$

- e is $\underline{\text{p-cocartesian}}$ if

$$\begin{tikzcd} \mathcal{D}_{d_1/} \arrow[d] & \mathcal{D}_{e/} \arrow[l, "\simeq"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{D}_{d_0/} \arrow[d] \\ \mathcal{C}_{pd_1/} & \mathcal{C}_{pe/} \arrow[l, "\simeq"'] \arrow[r] & \mathcal{C}_{pd_0/} \end{tikzcd} \in Cat.$$



"pullback across categories"

- e is $\underline{\text{p-cartesian}}$ if

$$\begin{tikzcd} \mathcal{D}_{/e} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{D}_{/d_1} \arrow[d] \\ \mathcal{C}_{/pe} \arrow[r] & \mathcal{C}_{/pd_1} \end{tikzcd}$$

- p is a cocart. fib if

$$\begin{tikzcd} \forall d \arrow[r, dashed, "\exists \bar{e}"] \arrow[d, mapsto] & \bar{c} \arrow[d, mapsto] \\ pd \arrow[r, "\forall e"'] & \forall c \end{tikzcd}$$

Cocart. transport exists

Cart fib ...

Rem e: p-cart $\Leftrightarrow$ $\forall d \in \mathcal{D}$

$$\begin{tikzcd} Map_{\mathcal{D}}(d_1, d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Map_{\mathcal{D}}(d_0, d) \arrow[d] \\ Map_{\mathcal{C}}(pd_1, pd) \arrow[r] & Map_{\mathcal{C}}(pd_0, pd) \end{tikzcd}$$

Rem • (co)cart edges / fibrations satisfy "pasting law of pullback" type stability properties. (stack two square diagrams)

- cocart fibs are closed under pullback & Fun($\mathcal{A}$, −) $\forall \mathcal{A} \in Cat$
- equivalences are cocart

Def A $\overset{\text{left}}{\text{right fibration}}$ is a $\overset{\text{co}}{}$cartesian fibration $p : \mathcal{D} \to \mathcal{C}$ satisfying one of the following equiv. conditions

(1) p: conservative

(2) $\forall X \in \mathcal{C}$, $\{X\} \times_{\mathcal{C}} \mathcal{D} \in \mathrm{Ani}$

(3) $\forall$ mor in $\mathcal{D}$ is $\overset{\text{co}}{}$Cartesian

ex •

$$\begin{tikzcd} \mathcal{C}^{\Delta^1} \arrow[d, "ev_1"] \\ \mathcal{C} \end{tikzcd}$$

is cocart fib

$$\begin{tikzcd} d \arrow[r, dashed, "="] \arrow[d, "g"'] & d \arrow[d, dashed, "f \circ g"] \\ c_0 \arrow[r, "f"'] & c_1 \end{tikzcd}$$

$$\begin{tikzcd} c_0 \arrow[r, "f"'] & c_1 \end{tikzcd}$$

cartesian edge = cartesian sq

• ev_0 is cart fib

More generally:

$$\begin{tikzcd} \mathcal{A} \vec{\times}_{\mathcal{C}} \mathcal{B} \arrow[r] \arrow[d] \arrow[rr, bend left, "\text{cocart}"] \arrow[dd, bend right, "\text{Cart}"'] \arrow[rd, phantom, "\lrcorner"] & \mathcal{C} \vec{\times}_{\mathcal{C}} \mathcal{B} \arrow[r] \arrow[d] \arrow[dd, bend right] \arrow[rd, phantom, "\lrcorner"] & \mathcal{B} \arrow[d] \\ \mathcal{A} \vec{\times}_{\mathcal{C}} \mathcal{C} \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[rd, phantom, "\lrcorner"] & \mathcal{C}^{\Delta^1} \arrow[r, "ev_1"'] \arrow[d, "ev_0"] & \mathcal{C} \\ \mathcal{A} \arrow[r] & \mathcal{C} & \end{tikzcd}$$

e.g.

$$\begin{tikzcd} & \mathcal{C}_{/x} \arrow[d, "\text{left fib}"] \\ \mathcal{C}_{y/} \arrow[r, "\text{right fib}"'] & \mathcal{C} \end{tikzcd}$$

cocart fib & functor which pres. cocart edges

$$\text{cf} \quad \begin{tikzcd} \mathrm{Cat}^{\text{cocart}}_{/\mathcal{C}} \arrow[r, shift right, hook] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_{/\mathcal{C}} : F \arrow[l, shift right] \end{tikzcd}$$

Thm

$$\begin{tikzcd} \mathrm{Cat}^{\text{cocart}}_{/\mathcal{C}} \arrow[r, shift left, "St"] & \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}) \arrow[l, shift left, "\int^{\text{cocart}}", "\simeq"'] \\ \mathrm{Cat}^{\text{lfib}}_{/\mathcal{C}} \arrow[u, phantom, "\subset" description, "\text{full}"'] \arrow[r, leftrightarrow, "\sim"] & \mathrm{Fun}(\mathcal{C}, \mathrm{Ani}) \arrow[u, phantom, "\subset" description] \\ \mathcal{C}_{x/} \arrow[u, phantom, "\in" description] \arrow[r, leftrightarrow] & X \arrow[u, phantom, "\in" description] \end{tikzcd}$$

,

$$\begin{tikzcd} \mathrm{Cat}^{\text{cart}}_{/\mathcal{C}} \arrow[r, shift left, "St"] & \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}) \arrow[l, shift left, "\int^{\text{cart}}", "\simeq"'] \\ \mathrm{Cat}^{\text{rfib}}_{/\mathcal{C}} \arrow[u, phantom, "\subset" description] \arrow[r, leftrightarrow, "\sim"] & \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ani}) \arrow[u, phantom, "\subset" description] \\ \mathcal{C}_{/X} \arrow[u, phantom, "\in" description] \arrow[r, leftrightarrow] & X \arrow[u, phantom, "\in" description] \end{tikzcd}$$

.

Rmk • functoriality in $\mathcal{C}$:

$$\begin{tikzcd} \mathcal{C} & \mathrm{Cat}^{\text{cocart}}_{/\mathcal{C}} \arrow[r, leftrightarrow, "\sim"] & \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}) \\ \mathcal{D} \arrow[u, leftarrow, "f"] & \mathrm{Cat}^{\text{cocart}}_{/\mathcal{D}} \arrow[u, "f^*"'] \arrow[r, leftrightarrow, "\sim"] & \mathrm{Fun}(\mathcal{D}, \mathrm{Cat}) \arrow[u, "- \circ f"'] \end{tikzcd}$$

• The universal lfib & cocart fib

$$\begin{tikzcd} \mathrm{Ani}_* \arrow[r] \arrow[d] \arrow[rd, phantom, "\lrcorner"] & \mathrm{Cat}_{*//} \arrow[d] & (\mathcal{C}, c) \arrow[l, bend right] \arrow[d, "{(f,\, \underbrace{fc \to d})}"] \\ \mathrm{Ani} \arrow[r] & \mathrm{Cat} & (\mathcal{D}, d) \end{tikzcd}$$

lax pointed

exercises (1) · Show $\mathrm{Gpd}_1^{\mathrm{str}} \hookrightarrow \mathrm{Cat}_1^{\mathrm{str}} \xhookrightarrow[N]{} \mathrm{sSet}$ fully faithful, (or at least you can reconstruct a category from its nerve)

· Show the horn-filling characterization of the essential image

(2) If $\mathcal{C} \in \mathrm{Kan}\text{-}\mathrm{Cat}$, show that

$$\begin{tikzcd}
\Lambda^3_1 \arrow[r] \arrow[d, hook] & N_{hc}(\mathcal{C}) \\
\Delta^3 \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

(this will indicate the proof for the general case)

(3) From the definitions, check left adjoints preserve colim
right adjoints preserve lim

(4) prove the following formula if I didn't.

Prop $\mathcal{C} \in \mathrm{Cat}_\infty$.

(i) $F : \mathcal{C} \to \mathrm{Ani} \rightsquigarrow \operatorname{colim} F = \left| \int F \right|$, $\lim F \simeq \mathrm{Map}_{/\mathcal{C}}(\mathcal{C}, \int F)$ $(\simeq \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \int F))$

(ii) $F : \mathcal{C} \to \mathrm{Cat} \rightsquigarrow \operatorname{colim} F = \left(\int F\right)[\mathrm{cocart}^{-1}]$, $\lim F \simeq \underline{\mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{C}}(\mathcal{C}, \int F)}$ cocart sections

$$\begin{tikzcd}
\rightsquigarrow & \int F \arrow[d, "\mathrm{cocart}"] \\
& \mathcal{C}
\end{tikzcd}$$

Rmk (ii) specializes to (i) because ∀mor in left fib is cocart.

(5) Construct the functor $\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Ani}$ if I didn't.
(or よ $: \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ani})$)

prove よ is limit-preserving.

(6) Assume the Yoneda lemma that よ: fully faithful.

· Show that if $f : \mathcal{C} \to \mathcal{D}$ has a local left / right adjoint for $\forall d \in \mathcal{D}$, they assemble into a functor $f^L, f^R : \mathcal{D} \to \mathcal{C}$.
and moreover show that these are uniquely determined from f.

· Show the equivalence to another definition of adj

(7) Suppose $\operatorname{colim}_{\lambda \in \Lambda} \mathcal{C}_\lambda \xrightarrow[i_\lambda]{\sim} \mathcal{C}$ and $\mathcal{C} \xrightarrow{f} \mathcal{D}$ (or assume $\mathcal{D} = \mathrm{Ani}$)

Prove · $\lim_{\mathcal{C}} f \xrightarrow{\sim} \lim_{\lambda} \lim_{\mathcal{C}_\lambda} f \circ i_\lambda$

· $\operatorname{colim}_{\lambda} \operatorname{colim}_{\mathcal{C}_\lambda} f \circ i_\lambda \longrightarrow \operatorname{colim}_{\mathcal{C}} f$. (for this sub-ex : localization is cofinal)