exercises

(1) Use the formula of (co)limits in Ani to show that $\text{よ}: \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is limit-preserving

(2) For $(\mathcal{C}, W) \in \mathrm{RelCat}_\infty$, show $\mathcal{C} \to \mathcal{C}[W^{-1}]$ is cofinal & coinitial.

(3) Suppose $\operatorname*{colim}_{\lambda \in \Lambda} \mathcal{C}_\lambda \xrightarrow[i_\lambda]{\sim} \mathcal{C}$ and $\mathcal{C} \xrightarrow{f} \mathcal{D}$ (or assume $\mathcal{D} = \mathrm{Ani}$)

Prove
- $\lim_{\mathcal{C}} f \xrightarrow{\sim} \lim_\lambda \lim_{\mathcal{C}_\lambda} f \circ i_\lambda$
- $\operatorname*{colim}_\lambda \operatorname*{colim}_{\mathcal{C}_\lambda} f \circ i_\lambda \longrightarrow \operatorname*{colim}_{\mathcal{C}} f$. (for this sub-ex : localization is cofinal)

(4) Show $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})^{op} \simeq \mathrm{Fun}^L(\mathcal{D}, \mathcal{C})$

(possibly hard.) $\simeq \mathrm{Cat}^{\text{cart \& cocart}}_{/\Delta^1} \underset{\mathrm{Cat}_{/\partial\Delta^1}}{\times} \underset{\swarrow(\mathcal{D}, \mathcal{C})}{*}$

(5) Show a right adjoint is cofinal (& left adjoint is coinitial)

(6) Show $\Delta^{inj} \hookrightarrow \Delta$ is coinitial.

$\quad$ mor : $[n] \underset{inj}{\longrightarrow} [m]$

(7) Show $\mathcal{C}$: κ-filtered $\iff$ $\mathrm{diag}: \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ is cofinal for any κ-small I.

(8) Show $\Delta \xrightarrow{\delta} \Delta \times \Delta$ is cofinal. (if you like, you may use the fact that $\Delta \hookrightarrow \mathrm{Cat}$ is dense)

(9) Consider $\mathcal{C} \overset{f}{\hookrightarrow} \mathcal{D}$ fully faithful, where $\mathcal{D}$ has κ-filt colim, $\mathcal{C}$ small

$$\rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r, hook, "f"] \arrow[d] & \mathcal{D} \\ \mathrm{Ind}_\kappa(\mathcal{C}) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

So that $\mathrm{Ind}_\kappa(\mathcal{C}) = \left\{ \begin{array}{c} \tilde{\mathcal{C}} \\ \downarrow \\ \mathcal{C} \end{array} \text{rfib. } \tilde{\mathcal{C}} : \text{filtered} \right\} \subset \mathcal{P}(\mathcal{C}) \simeq \mathrm{Cat}^{rfib}_{/\mathcal{C}}$

(1) if $\mathcal{C} \subset \mathcal{D}^\kappa$, $\overline{f}$ is fully faithful

(2) if moreover $\mathcal{C}$ generates $\mathcal{D}$ under κ-filt colim (i.e. the κ-filt colim closure of $\mathcal{C}$ is $\mathcal{D}$) then $\overline{f}$: equiv.

$X = \operatorname{colim} \text{よ} x_i$

(similar claims are true for $\mathcal{P}(\mathcal{C})$, $\mathcal{P}_\Sigma(\mathcal{C})$)

(10) Use (9) to recognize your favorite category as an Ind-completion

(11) Show that $\mathcal{C}$ is stable iff $\mathcal{C} \underset{\text{diag}}{\overset{\text{colim}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Fun}(J, \mathcal{C})$ has further left adjoint for $J = \phi, *, \Lambda^2_0 = \ulcorner$. (equivalently for all finite I)