Outline $\boxed{1}$ Categorical spectra   jt w/ Tim Campion

$\boxed{2}$ Review : tensor product of spectra

$\boxed{3}$ tensor product of categorical spectra

($\boxed{4}$ future direction)

Convention Everything is homotopical unless stated otherwise

"∞-" is implicit $\left(\begin{array}{l} \text{e.g. } n\text{-category} = (\infty,n)\text{-category} \quad (n\mathrm{Cat}) \\ \omega\text{-category} = (\infty,\infty)\text{-category} \\ \text{category} = (\infty,1)\text{-Category} \end{array} \right)$

$\boxed{1}$ Categorical spectra

Ref : Higher Quasicoherent Sheaves by Stefanich

Review : $\mathcal{S}$: cat of spaces

$$Sp = \lim\left(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right) \quad \text{cat of spectra}$$

· homotopy hypothesis

· n-cat

classical	homotopy theory	higher category theory
set = (0,0)-Cat, Set	spaces = $(\infty,0)$-Cat, $\mathcal{S}$	ω-Categories = (∞,∞)-Cat, $\omega\mathrm{Cat}$
$\mathrm{Set} \underset{\text{forget}}{\overset{\text{Free}}{\rightleftarrows}} \mathrm{Ab}$	$\mathcal{S} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} Sp$	$\omega\mathrm{Cat} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathrm{CatSp}$
Ab	spectra Sp	cat. sp CatSp
	$Sp \underset{B^\infty}{\overset{\Omega^\infty}{\rightleftarrows}} \mathrm{CMon}^{gp}(\mathcal{S})$	$\mathrm{CatSp} \underset{B^\infty}{\overset{\Omega^\infty}{\rightleftarrows}} \mathrm{CMon}(\omega\mathrm{Cat})$
	$\mathrm{CMon}^{gp}(\mathcal{S})$ grouplike $\mathbb{E}_\infty$-space	$\mathrm{CMon}(\omega\mathrm{Cat})$ symmetric monoidal ω-cats

$$0\mathrm{Cat} = \mathcal{S}, \quad 1\mathrm{Cat} = \mathrm{Cat}. \quad (n+1)\mathrm{Cat} = (n\mathrm{Cat})\text{-}\mathrm{Cat}$$

$$\rightsquigarrow \begin{tikzcd} (n+1)\mathrm{Cat} \arrow[r, shift left=3] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] \arrow[r, shift right=3, "\mathrm{core}_n"'] & n\mathrm{Cat} \arrow[l, hook'] \end{tikzcd}$$

core_n ← forget noninvertible $(n+1)$-morphisms

Def $\bullet$ $\omega\mathrm{Cat} := \lim\left(\cdots \to \overset{\color{red}{X_{n+1}}\,\rotatebox{90}{$\in$}}{(n+1)\mathrm{Cat}} \xrightarrow[\mathrm{core}_n]{} \overset{\color{red}{X_n}\,\rotatebox{90}{$\in$}}{n\mathrm{Cat}} \to \cdots \to \mathcal{S} \right)$

$\rightsquigarrow (\omega\mathrm{Cat})\text{-}\mathrm{Cat} \simeq \omega\mathrm{Cat}$

$\bullet$
$$\begin{tikzcd}
\omega\mathrm{Cat}_* \arrow[rr, shift left, "\Sigma"] \arrow[dr, shift right, "\mathrm{free}"'] & & \omega\mathrm{Cat}_* \arrow[ll, shift left, "\Omega"] \arrow[dl, shift left, "\mathrm{End}(*)"] \\
& \mathrm{Mon}(\omega\mathrm{Cat}) \arrow[ul, shift right, "\mathrm{forget}"'] \arrow[ur, shift left, "B"] &
\end{tikzcd}$$

$\underset{\underset{\mathrm{id}}{\rotatebox{90}{$\in$}}}{\mathrm{End}_X(x)} \longleftarrow\!\shortmid (X, x)$

ex. $\Sigma S^0 = B\mathbb{N} =: \vec{S}^1$

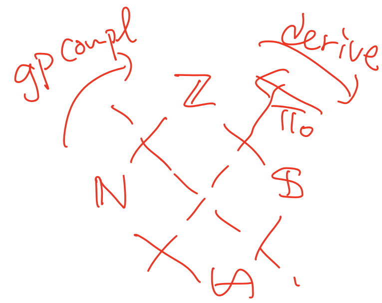
$O(\infty)$-action on inverted $\}$ dualizable

$\rightsquigarrow$ $\color{red}{\mathrm{CatSp} := \lim\left(\cdots \xrightarrow{\Omega} \omega\mathrm{Cat}_* \xrightarrow{\Omega} \omega\mathrm{Cat}_* \right)}$

⑰ examples

ex
$$\begin{tikzcd}
\mathrm{CatSp} = \lim\big(\cdots \arrow[r, "\Omega"] & \omega\mathrm{Cat}_* \arrow[r, "\Omega"] & \omega\mathrm{Cat}_* \big) \\
\mathrm{Sp} = \lim\big(\cdots \arrow[r, "\Omega"'] \arrow[u, hook] & \mathcal{S}_* \arrow[r, "\Omega"'] \arrow[u, hook] & \mathcal{S}_* \big) \arrow[u, hook]
\end{tikzcd}$$

$\left(\overset{\shortparallel}{\mathrm{Aut}} \text{ of base pt} \right)$



ex
$$\begin{tikzcd}
\mathrm{CMon}(\omega\mathrm{Cat}) \arrow[r, hook, "\color{red}{B^\infty}"] & \mathrm{CatSp} \\
\omega\mathrm{Cat} \arrow[u, "\mathrm{Free}"] \arrow[ur, "\Sigma^\infty_+"'] &
\end{tikzcd}$$

$\Sigma^\infty_+(*) = B^\infty \underset{\underset{\text{Free SMCat on }*}{\uparrow}}{\mathrm{Fin}^{\simeq}}$ $\color{red}{\text{“}\mathbb{S}\text{”}}$

cf. Baez–Dolan delooping hypothesis :

$\left(\{ \underbrace{\mathbb{E}_n\text{-mon.}}_{\color{red}{\uparrow \text{ n times deloopable}}}\ k\text{-Cat} \} \underset{B}{\overset{\Omega}{\rightleftarrows}} \{ \mathbb{E}_{n-1}\text{-mon } (k+1)\text{-Cat w/ single 0-mor} \} \right.$

$\underset{B}{\overset{\Omega}{\rightleftarrows}} \{ \mathbb{E}_{n-2}\text{-mon } (k+2)\text{-Cat w/ single } 0,1\text{-mor} \}$

$\rightleftarrows \cdots \underset{B}{\overset{\Omega}{\rightleftarrows}} \{ (n+k)\text{-Cat w/ single } 0, 1, \ldots, (n-1)\text{-mor} \}$

$$\begin{tikzcd}
\mathrm{CMon}^{gp}(\mathcal{S}) \simeq \mathrm{Sp}^{cn} \arrow[r, hook, "B^\infty"] \arrow[d, hook] & \mathrm{Sp} \arrow[d, hook] \\
\mathrm{CMon}(\omega\mathrm{Cat}) \arrow[r, hook, "B^\infty"] & \mathrm{CatSp}
\end{tikzcd}$$

$\color{red}{\text{stability without grouplike condition}}$

ex $A \in \mathrm{CAlg}(\mathrm{Sp})$

$$\begin{aligned}
\Rightarrow 0\text{-}\mathrm{Mod}_A &= \Omega^\infty A && \ni 1 \\
1\text{-}\mathrm{Mod}_A &= \mathrm{Mod}_A(\mathrm{Sp}) && \ni A \\
2\text{-}\mathrm{Mod}_A &= \mathrm{Mod}_{\mathrm{Mod}_A}(\mathrm{Pr}^{L,\omega}) && \ni \mathrm{Mod}_A \\
&\vdots
\end{aligned}$$

$\overset{\text{groupoid core}}{\rightsquigarrow}$ $A^\times$, $\mathrm{Pic}(A)$, $\mathrm{Br}(A)$

$\rightsquigarrow$ categorical spectrum $\{n\mathrm{Mod}_A\}$

ex $\mathcal{C}$: sym mon cat w/ good relative $\otimes$

$\rightsquigarrow \mathrm{Morita}_n(\mathcal{C})$ higher Morita cat of $\mathbb{E}_n$-alg :

obj : $\mathbb{E}_n$-alg in $\mathcal{C}$

mor $A \to B$: $\mathbb{E}_{n-1}$-alg in ${}_A\mathrm{BMod}_B(\mathcal{C})$

2-mor $M \to N$: $\mathbb{E}_{n-2}$-alg in ${}_M\mathrm{BMod}_N({}_A\mathrm{BMod}_B(\mathcal{C}))$

$\vdots$

$\rightsquigarrow \mathrm{End}_{\mathrm{Morita}_{n+1}(\mathcal{C})}(1_{\mathcal{C}}) \simeq \mathrm{Morita}_n(\mathcal{C})$

$\mathcal{S}$

ex $\mathcal{C}$ $(\simeq \mathcal{S})$: Cat w/ finite limits

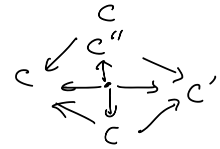
σ-model ?

$\mathrm{Hom}_{n\mathrm{Corr}}(c,c') = (n-1)\mathrm{Corr}(\mathrm{Hom}_{2\mathrm{Corr}}(c,c'))$

- $\underline{\mathrm{Corr}}(\mathcal{C}) = \{(n\mathrm{Corr}(\mathcal{C}), 1_{\mathcal{C}})\}$

  $\uparrow$ n-cat of correspondence

  - obj = obj $\mathcal{C}$
  - mor(c,c') = $c \leftarrow c'' \rightarrow c'$
  - 2-mor =



  $\vdots$

  $\rightsquigarrow \mathrm{End}_{(n+1)\mathrm{Corr}(\mathcal{C})}(1_{\mathcal{C}}) = n\mathrm{Corr}(\mathcal{C})$

- $\{\omega\mathrm{Corr}(\mathcal{C})\}$: 1-periodic

  $(\mathrm{End}_{\omega\mathrm{Corr}(\mathcal{C})}(1_{\mathcal{C}}) = \omega\mathrm{Corr}(\mathcal{C}))$

Thm(Stefanich) $\{n\mathrm{QCoh} : n\mathrm{Corr}(\mathrm{Prestk}) \longrightarrow n\mathrm{Pr}^L_{st}\}$ mor ~~of~~ categorical spectra

$\boxed{2}$ Review : $\otimes$ of Sp

Modern construction of the smash product of spectra

~~Digression~~ ($\otimes$ of presentable categories)

- A $\overset{\text{(large)}}{\text{cat}}$ $\mathcal{C}$ is presentable $\iff$ $\exists \mathcal{D}$ small, $\mathcal{C} \underset{\underset{\text{accessible}}{\uparrow}}{\overset{\exists}{\rightleftarrows}} \mathcal{P}(\mathcal{D}) := \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{S})$
- $\mathrm{Pr}^L \subset \mathrm{CAT}$ pres. cats & colim pres functors "LFun"

  $\left(\begin{array}{c} \Updownarrow \text{AFT} \\ \text{left adj} \end{array} \quad . \quad \mathrm{Pr}^R = (\mathrm{Pr}^L)^{op} \right)$

- $\mathcal{P} : \mathrm{Cat} \longrightarrow \mathrm{Pr}^L$ is the "free cocompletion" functor :

$$\mathrm{LFun}(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

Thm (Lurie) • Pr^L admits a unique closed sym. mon. str. making $\mathcal{P}$ (strong) monoidal

- The internal hom is LFun
- The unit is $\mathcal{S} = \mathcal{P}(*)$

(i.e. $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ universal 2-variables fcr which distributes colimts in each var.

Back to Sp:

$\mathrm{Pr}^L, \mathrm{Pr}^R \hookrightarrow \mathrm{CAT}$ commutes w/ (small) limits.

$$\rightsquigarrow \mathrm{Sp} = \lim_{\text{in } \mathrm{Pr}^R} \left(\cdots \longrightarrow \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right)$$

$$\mathcal{S} \xrightarrow{\Sigma^\infty_+} \quad = \operatorname*{colim}_{\text{in } \mathrm{Pr}^L} \left(\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \longrightarrow \cdots \right) \quad \in \mathrm{Pr}^L$$

$$\mathcal{S} \xrightarrow{(-)_+} \mathcal{S}_*$$

Thm ① $\mathcal{S} \xrightarrow{\Sigma^\infty_+} \mathrm{Sp}$ is idempotent (in Pr^L)
(Lurie)

i.e. $\mathcal{S} \otimes \mathrm{Sp} \xrightarrow{\Sigma^\infty_+ \otimes \mathrm{id}} \mathrm{Sp} \otimes \mathrm{Sp}$: equiv.

② $\mathrm{Sp} \otimes - \circlearrowright \mathrm{Pr}^L$ localization w/ image Pr^L_{st}, closed under $\otimes$

③ $\mathrm{Sp} \in \mathrm{Pr}^L_{st}$: $\otimes$-unit $\Rightarrow$ $\mathrm{Sp}^\otimes \in \mathrm{CAlg}(\mathrm{Pr}^L_{st}) \hookrightarrow \mathrm{CAlg}(\mathrm{Pr}^L_{st})$, $\mathrm{Pr}^L_{st} = \mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^L)$

Characterized by "closed + $\Sigma^\infty_+ : \mathcal{S}^\times \longrightarrow \mathrm{Sp}^\otimes$ symm. monoidal" $\left(\begin{array}{c} \text{cf.} \\ \mathrm{Set}^\times \to \mathrm{Ab}^\otimes \end{array} \right)$

$\boxed{3}$ $\otimes$ on CatSp

<u>Naive strategy</u> $\mathrm{CatSp} = \underset{\text{in } \mathrm{Pr}^L}{\mathrm{colim}}\left(\omega\mathrm{Cat}_* \xrightarrow{\Sigma} \omega\mathrm{Cat}_* \to \cdots\right)$

- $\rightsquigarrow$ show $\omega\mathrm{Cat} \xrightarrow{\Sigma^\infty_+} \mathrm{CatSp}$: idempotent in $\mathrm{Mod}_{\omega\mathrm{Cat}}(\mathrm{Pr}^L)$
  Cart prod
- $\rightsquigarrow$ same as above ---

but $\omega\mathrm{Cat}_* \xrightarrow{\Sigma} \omega\mathrm{Cat}_*$ is not a $\omega\mathrm{Cat}$-module hom

$X \wedge Y \longmapsto \Sigma(X \wedge Y)$ roughly:

$\quad \hookrightarrow X \wedge (\Sigma Y)$ Category level of $\left.\begin{array}{l} X : n \\ Y : m \end{array}\right\} \rightsquigarrow \begin{array}{l} \Sigma(X\wedge Y) : \max(n,m)+1 \\ \quad\quad \neq \\ X \wedge (\Sigma Y) : \max(n, m+1) \end{array}$

<u>What's wrong with Σ on $\omega\mathrm{Cat}_*$?</u>

- $S_* \xrightarrow{\Sigma} S_* \quad \simeq (-)\wedge S^1 \simeq S^1 \wedge (-)$
- $\omega\mathrm{Cat}_* \longrightarrow \omega\mathrm{Cat}_* \quad \simeq (-)\otimes \vec{S}^1 \neq \vec{S}^1 \otimes (-)$

  ((lax) Gray smash product)

(lax) Gray tensor product : biclosed monoidal str. on $\omega\mathrm{Cat}$
(partially conjectual?)

- $(n\text{-Cat}) \otimes (m\text{-cat})$: $(n+m)$-cat

  <u>ex</u>
$$\begin{tikzcd} 0 \arrow[d] \\ 1 \end{tikzcd} \otimes \; 0 \to 1 \; = \; \begin{tikzcd} 00 \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & 01 \arrow[d] \\ 10 \arrow[r] & 11 \end{tikzcd}, \quad \square^n \otimes \square^m = \square^{n+m}$$

  "fully lax" n-Cube

- Unit $= *$
- $\mathrm{Map}(X \otimes Y, Z) \simeq \mathrm{Map}(X, \mathrm{Fun}^{\mathrm{lax}}(Y, Z))$

  obj : functors $Y \to Z$
  mor : lax nat tr.
  $\vdots$

- $\omega\mathrm{Cat}^{\times} \xrightarrow{\mathrm{id}} \omega\mathrm{Cat}^{\otimes}$: lax monoidal $(X \otimes Y \to X \times Y)$

$$\begin{tikzcd} \mathrm{Hom}_X(x_0,x_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{lax}(\square^1, X) \arrow[d] \\ * \arrow[r, "{(x_0,x_1)}"'] & \mathrm{Fun}^{lax}(\partial\square^1, X) = X \times X \end{tikzcd}$$

in $\omega\mathrm{Cat}$

If we use $\mathrm{Fun}(\square^1, X)$ instead

$$\frac{\square^1 \longrightarrow \mathrm{Fun}(\square^1, X)}{\times \quad = \left[\overrightarrow{\simeq}\right] \longrightarrow X}$$

ignores noninvertible 2-cells of X
(& higher)

$\rightsquigarrow$ let
$$\begin{tikzcd} \partial\square^1 \arrow[r] \arrow[d] & \square^1 \arrow[d] \\ * \arrow[r] & \vec{S}^1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \quad (= BN)$$

and write $\circledast$ for the induced mon. str. on $\omega\mathrm{Cat}_*$

<u>Notation</u> $A = \omega\mathrm{Cat}_*^{\circledast} \in \mathrm{Alg}(\mathrm{Pr}^L)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{V}$

Then: $\Omega X = \mathrm{End}_X(x) = \mathrm{Fun}_*^{lax}(\vec{S}^1, X)$

$$\rightsquigarrow \quad A \underset{\Omega \, = \, \mathrm{Fun}_*^{lax}(\vec{S}^1, -)}{\overset{\Sigma \, = \, - \circledast \vec{S}^1}{\underset{\perp}{\rightleftarrows}}} A$$

Σ: naturally in $\mathrm{LMod}_A(\mathrm{Pr}^L)$

<u>Q</u>

Is $\mathrm{colim}\left(A \xrightarrow{-\circledast \vec{S}^1} A \longrightarrow \cdots\right) \in \mathrm{LMod}_A(\mathrm{Pr}^L)$ idempotent?

<u>Problem</u> Doesn't make sense. $\mathrm{LMod}_A(\mathrm{Pr}^L)$ only $\mathbb{E}_0$

<u>but</u> $\mathrm{BMod}_A(\mathrm{Pr}^L)$: $\mathbb{E}_1$

<u>Q1</u>

$$\begin{tikzcd}
 & \mathrm{End}_{\mathrm{BMod}_A}(A) \arrow[d, "\text{forget}"] \arrow[r, phantom, "\simeq"] & HH^\bullet_V(A) \arrow[d] & \\
\Sigma \in \arrow[ur, bend left, "\exists \text{ lift?}"] & \mathrm{End}_{\mathrm{LMod}_A}(A) \arrow[r, phantom, "\simeq"] & A^{rev} & \ni \vec{S}^1
\end{tikzcd}$$

<u>Q2</u> $\Sigma^\infty_+ : \omega\mathrm{Cat} \longrightarrow \mathrm{CatSp}$: idempotent in $\mathrm{BMod}_{\omega\mathrm{Cat}}(\mathrm{Pr}^L)$?

$$HH^\bullet_V(A) \simeq \mathrm{Tot}\left(A \rightrightarrows [A,A] \mathrel{\substack{\to\\\to\\\to}} [A\otimes A, A] \mathrel{\substack{\to\\\to\\\to\\\to}} \cdots\right) \simeq \mathrm{End}_{\mathrm{End}_{V\text{-}\mathrm{Cat}}(BA)}(\mathrm{id})$$

$a \mapsto -\otimes a$, $\quad a \mapsto a \otimes -$

$\Phi \mapsto -\otimes\Phi(-)$, $\quad \Phi \mapsto \Phi(-\otimes-)$, $\quad \Phi \mapsto \Phi(-)\otimes -$ $\quad \cdots$

[AKA $Z(A)$: (Drinfeld) center of A]

<u>data</u>: ⓪ object $a \in A$

① nat. isom $\Phi_x : a\otimes x \xrightarrow{\sim} x \otimes a$

② nat. htpy

$$\begin{tikzcd}
 & x\otimes a\otimes y \arrow[dr, "x\otimes\Phi_y"] & \\
a\otimes x\otimes y \arrow[ur, "\Phi_x\otimes y"] \arrow[rr, "\Phi_{x\otimes y}"'] \arrow[rr, phantom, shift left=3, "\Psi_{x,y}"] & & x\otimes y\otimes a
\end{tikzcd}$$

③ invertible 3-cell

$$\begin{tikzcd}
 & x\otimes a\otimes y\otimes z \arrow[dd] \arrow[dr] & \\
a\otimes x\otimes y\otimes z \arrow[ur] \arrow[rr] \arrow[dr] & & x\otimes y\otimes z\otimes a \\
 & x\otimes y\otimes a\otimes z \arrow[ur] &
\end{tikzcd}$$

$\vdots$

Q1 : false, $X \otimes \vec{S}^1 \not\simeq \vec{S}^1 \otimes X$

but $X \otimes \vec{S}^1 \simeq \vec{S}^1 \otimes X^\circ$

$\left(\begin{array}{l} (-)^\circ : \omega Cat \longrightarrow \omega Cat \\ \text{total dual (monoidal)} \end{array} \right.$

sketch : $\exists$ pushout diagrams

$$\begin{tikzcd} X \otimes \partial \square^1 \arrow[r] \arrow[d] & X \otimes \square^1 \arrow[d] \\ \partial \square^1 \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & SX \end{tikzcd} \qquad \begin{tikzcd} \partial \square^1 \otimes X^\circ \arrow[r] \arrow[d] & \square^1 \otimes X^\circ \arrow[d] \\ \partial \square^1 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & SX \end{tikzcd}$$

SX ~ unreduced suspension $\begin{tikzcd} \bot \arrow[r, "X"] \arrow[loop below, "id"] & \top \arrow[loop below, "id"] \end{tikzcd}$

⇝ formally :

$$\begin{tikzcd} S* \arrow[r] \arrow[d, equal] & SX \arrow[r] \arrow[d, equal] & X \otimes \vec{S}^1 \arrow[d, "\cong"] \\ S* \arrow[r] & SX \arrow[r] & \vec{S}^1 \otimes X^\circ \end{tikzcd}$$

① $X \otimes \vec{S}^1 \simeq \vec{S}^1 \otimes X^\circ$

② $\begin{tikzcd} X \otimes Y \otimes \vec{S}^1 \arrow[r, "\sim"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\not\sim"] & \vec{S}^1 \otimes (X \otimes Y)^\circ \arrow[d, equal] \\ X \otimes \vec{S}^1 \otimes Y^\circ \arrow[r, "\sim"] & \vec{S}^1 \otimes X^\circ \otimes Y^\circ \end{tikzcd}$

③

⋮

Q. what is this structure?

$$\mathrm{Tot}\left(A \rightrightarrows [A,A] \rightrightarrows [A \times A, A] \rightrightarrows \cdots \right) \simeq \mathrm{Hom}_{\mathrm{End}_{V\text{-}Cat}(BA)}(id, (-)^\circ)$$

$a \in A$:
$a \longmapsto - \otimes a$
$a \longmapsto a \otimes (-)^\circ$

$\Phi \longmapsto - \otimes \Phi(-)$
$\Phi \longmapsto \Phi(- \otimes -)$
$\Phi \longmapsto \Phi(-) \otimes (-)^\circ$

$\cdots$

$$\mathcal{C}^\circ \overset{?}{\underset{D}{\xrightarrow{\sim}}} \mathcal{C}$$

Thm ① The data $(\vec{S}^1, (-)\circledast\vec{S}^1 \to \vec{S}^1\circledast(-)^\circ)$ lifts uniquely to $\mathrm{Hom}_{\mathrm{End}_{V\text{-Cat}}(BA)}(\mathrm{id}, (-)^\circ)$ (similarly for $\mathrm{Hom}((-)^\circ, \mathrm{id})$)

② $\vec{S}^2$ admits a canonical lift to $Z(A)$

③ $\mathrm{CatSp} = \underset{\text{in } \mathrm{Pr}^L}{\mathrm{colim}}(\omega\mathrm{Cat}_* \xrightarrow{\Sigma^2} \omega\mathrm{Cat}_* \xrightarrow{\Sigma^2} \cdots) \in \mathrm{BMod}_{\omega\mathrm{Cat}}(\mathrm{Pr}^L)$

is idempotent

$\rightsquigarrow$ CatSp admits a biclosed monoidal str.

(unit : $\Sigma^\infty_+ S^0 = B^\infty \mathrm{Fin}^\simeq$)

④ $\mathrm{CatSp}^\otimes = \omega\mathrm{Cat}_*^\circledast[\vec{S}^{1^{-1}}]$

... under construction ...

- $\omega\mathrm{Cat}_*^\circledast \longrightarrow \mathrm{CatSp}^\otimes$: monoidal characterize ?
- $\mathrm{BMod}_{\omega\mathrm{Cat}^\circledast}(\mathrm{Pr}^L) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathrm{BMod}_{\mathrm{CatSp}^\otimes}(\mathrm{Pr}^L)$ has a monoidal refinement

① Known: $\square \hookrightarrow \omega\mathrm{Cat}$ is dense [Campion]

full sub spanned by $\square^n = (\square^1)^{\otimes n}$

$\rightsquigarrow \square_+ \hookrightarrow \omega\mathrm{Cat}_*$ is dense.

full sub of $(\square^n)_+$

Coherence lives in the Aut space of

$$\mathrm{LFun}(\omega\mathrm{Cat}_* \otimes \cdots \otimes \omega\mathrm{Cat}_*, \omega\mathrm{Cat}_*) \ni (-)\owedge \cdots \owedge (-) \owedge \vec{S}^1$$

$$\downarrow$$

$$\mathrm{LFun}(\mathcal{P}(\square_+) \otimes \cdots \otimes \mathcal{P}(\square_+), \omega\mathrm{Cat}_*)$$

$$\wr\!|$$

$$\mathrm{Fun}(\underbrace{\square_+ \times \cdots \times \square_+}_{(1,1)\text{-cat}}, \omega\mathrm{Cat}_*) \ni \underbrace{(\square^{n_1}_+) \owedge \cdots \owedge (\square^{n_k}_+) \owedge \vec{S}^1}_{\cong\, (\square^{n_1} \otimes \cdots \otimes \square^{n_k})_+}$$

Now $X_+ \owedge \vec{S}^1 = B\,\mathrm{Free}_{\mathbb{E}_1}(X_+)$

$$= B\left(\coprod_{n\geq 0}^{\infty} X^{\times n}\right) \rightsquigarrow \text{can recover } X \text{ by indec} \circ \Omega$$

so $\mathrm{Aut}_{\omega\mathrm{Cat}_*}(X \owedge \vec{S}^1) = \mathrm{Aut}_{\omega\mathrm{Cat}_*}(B\,\mathrm{Free}_{\mathbb{E}_1}(X_+))$

$$\xrightarrow[\Omega]{\sim} \mathrm{Aut}_{\mathrm{Mon}(\omega\mathrm{Cat})}(\mathrm{Free}_{\mathbb{E}_1}(X_+)) \xrightarrow[\text{indec}]{\sim} \mathrm{Aut}_{\omega\mathrm{Cat}}(X)$$

but $\mathrm{Aut}(\square^n) = *$ $\rightsquigarrow$ no nontrivial coherence!

④ Key:

$$\begin{tikzcd} A \arrow[r, "\Sigma^2"] \arrow[d, "\Sigma^2"'] \arrow[dr, phantom, "\Swarrow"] & A \arrow[r] \arrow[d, "\Sigma^2"] & \cdots \arrow[r] & A \arrow[d, "\Sigma^2"] \\ A \arrow[r, "\Sigma^2"'] & A \arrow[r] & \cdots \arrow[r] & A \end{tikzcd}$$

Σ^2 : invertible

$\Uparrow$

Comes from $\mathrm{Aut}(\vec{S}^4) = \{\mathrm{id}\}$

$\boxed{4}$

- Compute $\otimes$ of examples
- Define & compute important invariants of algebra
  e.g. cotangent complexes, what is $L_{\mathbb{Z}/B^\infty \mathrm{Fin}^{\simeq}}$?
- $\vec{S}^1$: central up to $\overset{\text{fermionic?}}{\text{dual}}$, $\vec{S}^2$: $\overset{\text{bosonic?}}{\text{central}}$
  $\leftarrow\text{---}\rightarrow$ some "graded commutativity" ?
  formulate & do "commutative algebra"
- being a CatSp-module is a property of $\mathrm{wCat}^{\otimes}$-bimod.
  what does this imply for more general (partially) (op)lax (co)lim than $\Sigma \dashv \Omega$?

  relation to other proposed notions of stable n-cats?
- Reinterpret Connes–Consani's absolute geometry

$\vdots$

🤔😵‍💫🫠

Thank You!

$$\underset{\substack{\shortparallel \\ A}}{w\mathrm{Cat}_*^{\otimes}} \xrightarrow[\Sigma^2]{\substack{\Sigma \\ \shortparallel \\ \otimes \vec{S}^1}} w\mathrm{Cat}_*^{\otimes} \quad \in \mathrm{Hom}_{\mathrm{LMod}_A(V)}(A,A)$$

$V = \Pr^{L,\omega}$

$A \in \mathrm{Alg}(V)$

$$\uparrow$$

$$\mathrm{Hom}_{\mathrm{BMod}_A(V)}(A,A)$$

$$\mathrm{Hom}_{\mathrm{BMod}_A(V)}(A,A) \simeq \mathrm{End}_{\mathrm{End}_{\mathrm{Morita}(V)}(A)}(\mathrm{id}_A)$$

$\sim$

$$\simeq \mathrm{End}_{V\text{-}\mathrm{Prof}(P(BA))}(\mathrm{id})$$

$$\simeq \mathrm{End}_{\mathrm{End}_{V\text{-}\mathrm{Alg}\,\mathrm{brd}}(BA)}(\mathrm{id})$$

$$\simeq \mathrm{Tot}\left(A \rightrightarrows [A,A] \rightrightarrows [A\otimes A, A] \rightrightarrows \cdots\right)$$

$a \mapsto -\otimes a$

$a \mapsto a \otimes -$

$\mathrm{RMod} \rightleftarrows \mathrm{LMod}(\mathrm{RMod})$

$[n] \longmapsto (\Delta_{/[n]})^{op} \longmapsto \mathrm{Alg}_{\Delta^{op}_{/[n]}}(V)$

$[p]*[q]$

$\shortparallel$

BM $\to \Delta_{/[1]}$ $\quad [n] \to [1]$

$(0,0,\dots,0,1,\dots,1)$

$\downarrow$

Δ

$A \longrightarrow A$

free $\longrightarrow A\otimes A \longrightarrow A$

$A \otimes A \otimes A$

ev

$$V\text{-}\mathrm{Alg} \longrightarrow \mathrm{Morita}(V)$$

locally ff

$$\mathrm{BMod}_A(V) = \mathrm{LMod}_A(\mathrm{RMod}_A(V)) = V\text{-}\mathrm{Fun}(BA, V\text{-}\mathrm{Fun}(BA^{op}, V)) \simeq \mathrm{Hom}_{\mathrm{LMod}_V(A)}(P(BA), P(BA))$$

$$\simeq V\text{-}\mathrm{Prof}(P(BA), P(BA))$$

$$\mathrm{Tot}\left(A \underset{G}{\overset{F}{\rightrightarrows}} [A,A] \underset{G}{\overset{F}{\rightrightarrows}} [A\otimes A, A] \underset{G}{\overset{F}{\rightrightarrows}} \cdots\right)$$

$a \mapsto F(-)\otimes a$

$a \mapsto a \otimes G(-)$

$\Phi \mapsto F(-)\otimes \Phi(-)$

$\Phi(-\otimes -)$

$\Phi(-)\otimes G(-)$

$F(-)\otimes F(-)\otimes a \otimes G(-)\otimes G(-)\otimes G(-)$

$\Phi \qquad x\otimes a \otimes y \qquad \Phi$

$x \otimes y \otimes a \longrightarrow a \otimes x \otimes y$

Φ

$(-)\otimes \vec{S}^1 \simeq \vec{S}^1 \otimes (-)^{\circ}$ ~~total~~ dual coop

$$\vec{S}^2 \in \mathrm{End}_{\mathrm{End}_{V\text{-}\mathrm{Cat}}(BA)}(\mathrm{id}) \Longleftarrow \vec{S}^1 \in \mathrm{End}_{\mathrm{End}_{V\text{-}\mathrm{Cat}}(BA)}(\mathrm{id}, (-)^{\circ})$$

$\wr\shortparallel$

$$\mathrm{End}_{\mathrm{End}}((-)^{\circ}, \mathrm{id})$$

monoidal $BA \to BA$

$$\begin{tikzcd} BA \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & BA \end{tikzcd}$$

$$\mathrm{Nat}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F,G) = \int_{c\in\mathcal{C}} \mathcal{D}(F(c),G(c))$$

$$= \lim_{\Delta/\mathcal{C}} (\sim) \qquad \simeq \lim_{\mathrm{Tw}(-)}$$

Rune's paper

$\mathcal{C}$: ∞-Cat

$$\prod_{c\in\mathcal{C}} \mathcal{D}(F(c),G(c)) \rightrightarrows \prod_{\substack{\cancel{f:c\to c'\in\mathcal{C}}\\ \Delta^1\to\mathcal{C}}} \mathcal{D}(F(c),G(c'))$$

$\times$ $\mathrm{Hom}(-,-) \circlearrowleft$ -1

$$\mathrm{Hom}(-,-) \underset{??}{\overset{\text{"}-1\text{"}}{\rightleftarrows}} \mathrm{Hom}(-,(-)^o)$$

$$\mathrm{End}_{\mathrm{End}_{V\text{-Cat}}(BA)}(A) = \mathrm{Tot}\left(A \overset{(-)^o}{\rightrightarrows} [A,A] \overset{(-)^o}{\Rrightarrow} [A\otimes A, A] \Rrightarrow \cdots\right)$$

obstructions should be like

$$\begin{tikzcd}
* \arrow[r] \arrow[rr, bend left] & \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{LFun}^\omega(A\otimes\cdots\otimes A, A) \\
(-)\otimes\cdots\otimes(-)\otimes \vec{S}^1 \arrow[u] & \Delta^n \arrow[ur] &
\end{tikzcd}$$

$\mathrm{LFun}(A\otimes\cdots\otimes A, A) \supseteq \boxed{\mathrm{Fun}((\square_+^{\mathrm{full}})^{\times n}, A)}$

$S^0 \downarrow S_* \otimes \mathcal{C} \simeq \mathcal{C}_*$

- $\square^{\mathrm{full}} \hookrightarrow \omega\mathrm{Cat}$ dense
  full sub of $(\square^1)^{\otimes n}$

$\mathcal{P}(\square) \downarrow \omega\mathrm{Cat}$ $\rightsquigarrow$ $\square_+^{\mathrm{full}} \hookrightarrow \omega\mathrm{Cat}_*$ dense.
full sub of $(\square^1)_+^{\otimes n}$

$$\mathcal{P}(\square_+^{\mathrm{full}}) \underset{}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \omega\mathrm{Cat}_*$$

$\mathcal{C} \hookrightarrow \mathcal{D}$ dense
$\Downarrow$
$\{\mathrm{Free}_T(c)\}_{c\in\mathcal{C}} \hookrightarrow \mathrm{Alg}_T(\mathcal{D})$ dense

$$\mathrm{Aut}(\square_+^n \otimes \vec{S}^1) \simeq *$$
$$\text{"} \; B\,\mathrm{Free}(\square_+^n)$$

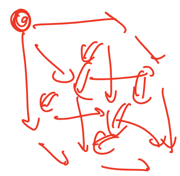
$$(X_{1+} \otimes \cdots \otimes X_{n+} \otimes \vec{S}^1) = (X_1 \wedge \cdots \wedge X_n)_+ \otimes \vec{S}^1$$

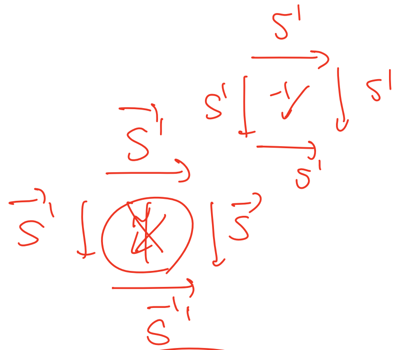
$$\mathrm{Aut}_{\omega\mathrm{Cat}_*}(B\,\mathrm{Free}(\square^n_+)) = \mathrm{Aut}_{\mathrm{Mon}(\omega\mathrm{Cat})}(\mathrm{Free}(\square^n_+)) \hookleftarrow \coprod_{k\geq 0} (\square^n)^{\times k}$$

$$\xrightarrow[\text{indec}]{\sim} \mathrm{Aut}_{\omega\mathrm{Cat}}(\square^n) \simeq *$$

$$\coprod_{k\geq 0}(\square^n)^{\times k} \rightsquigarrow_{\text{indec}} \square^n$$

$\mathbb{S} \in \mathrm{Sp}$





$$\mathrm{colim}\left(A \xrightarrow{x} A \xrightarrow{x} A \to \cdots\right) \xrightarrow{=} A_x$$

$$\begin{array}{ccccc} A & \xrightarrow{x} & A & \to \\ x\downarrow & x^{-1}\!\!\!\!/ & \downarrow x & // \\ A & \xrightarrow{x} & A & \to \end{array} \qquad A_x \xrightarrow{x} A_x$$

bimod hom str. on x

$$A[x^{-1}] = A_x$$

$$B^4\,\mathrm{Free}_{\mathbb{E}_4}(*)$$

$$A \xrightarrow[\vec{S}^4]{} A \qquad \pi_0\ \underset{\omega\mathrm{Cat}_*}{\overset{\mathrm{End}}{\mathrm{Aut}}}(\vec{S}^4) \simeq \underset{\mathrm{Alg}_{\mathbb{E}_4}(\omega\mathrm{Cat})}{\overset{\mathrm{End}}{\mathrm{Aut}}}(\mathrm{Free}_{\mathbb{E}_4}(*))$$

$$\simeq \mathrm{Map}_{\omega\mathrm{Cat}}(*, \mathrm{Free}_{\mathbb{E}_4}(*))$$

$$\pi_0 \simeq |\mathrm{Free}_{\mathbb{E}_4}(*)| \simeq \mathbb{N}$$

id

$$\cdots \xrightarrow{??} \mathrm{Fin} \to \mathbb{N}$$

lax Čech nerve & descent

$\mathbb{N} \otimes_{\mathrm{Fin}} \mathbb{N}$

$\mathbb{Z} \otimes_{\mathrm{Fin}} \mathbb{Z}$

Steenrod alg ?

how generating $\mathbb{H}\in \mathrm{CatSp}$?

$$\mathbb{N} \otimes_{\mathrm{Fin}} \mathbb{N} \longleftarrow \mathbb{N}$$

$$\pi_0\left(\mathbb{N} \otimes_{\mathrm{Fin}} \mathbb{N}\right) \qquad \pi_0 \mathbb{N}$$

n-cat

n-mor

Fam_d (X, ζ)-str

$$\mathrm{CatSp} \longrightarrow \mathrm{CatSp}$$

$\{\mathcal{C}_n\}$ $\mathrm{Fam}_n(\mathcal{C}_n)$

$$\Omega \mathrm{Fam}_n(\mathcal{C}_n) = \mathrm{Fam}_{n-1}(\Omega \mathcal{C}_n)$$

<u>Ent theory</u>

$$\mathrm{Sh}(X)$$

f^*

$\mathcal{P}(\mathcal{C})$-enriched category

$$\underline{V} = B^\infty\Big(\coprod_{n \geq 0} BU(n)\Big)$$

$$BAut(\mathbb{C}^n)$$

$B\mathrm{Aut}(\mathbb{F}_1^n)$