§1 CatSp

§2 $\otimes$

§3 stability

§4 appl. to TQFT

(w.ho.type of)
spaces / anima / ∞-grpd

§1 $\mathrm{Sp} := \lim\left(\cdots \longrightarrow \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*\right) = 0\mathrm{Cat}_*$ (in $\widehat{\mathrm{Cat}}$ or Pr^R)

$X = (X_n, X_n \xrightarrow{\sim} \Omega X_{n+1}) \in \mathrm{Sp}$, $\quad (X,x) \in \mathcal{S}_* \longmapsto \mathrm{End}_X(x) \in \mathcal{S}_*$ $\left(= \mathrm{Aut}_X(x) = [S^1, X]_*\right)$ $\neq$

$$\begin{tikzcd}
\mathrm{Sp} \arrow[d] & \mathcal{S}_* \arrow[r, "\Omega"] \arrow[d] & \mathcal{S}_* \arrow[d] \\
\mathrm{CatSp} & \infty\mathrm{Cat}_* \arrow[r, "\Omega"] & \infty\mathrm{Cat}_*
\end{tikzcd}$$

Def $\mathrm{CatSp} := \lim\left(\cdots \longrightarrow \infty\mathrm{Cat}_* \xrightarrow{\Omega} \infty\mathrm{Cat}_*\right)$

("directed ho.type")

$0 \leq n \leq \infty$

Notation $n\mathrm{Cat}$: $(\infty,1)$-cat of large (∞,n)-cat
implicit

(warning : $\mathrm{CatSp} \notin \infty\widehat{\mathrm{Cat}}$)

$n = \infty$: "inductive" equv. i.e. $\infty\mathrm{Cat} = \mathrm{colim}_n\, n\mathrm{Cat}$ in Pr^L

$$\begin{tikzcd}
\mathrm{Sp} \arrow[rr, shift right, "\Omega^\infty"'] \arrow[rr, phantom, "\perp"] \arrow[dr] \arrow[dd, shift left] & & \mathcal{S}_* \arrow[ll, shift right, "\Sigma^\infty_{\mathrm{Sp}}"'] \arrow[dd] \\
& \mathrm{CMon}^{gp}(\mathcal{S}) \simeq \mathrm{Sp}^{cn} \arrow[ul, bend left] \arrow[ur] \arrow[dd] & \\
\mathrm{CatSp} \arrow[uu, shift left, "(-)^{gp}"] \arrow[uu, phantom, "\dashv"] \arrow[rr, shift right, "\Omega^\infty"'] \arrow[rr, phantom, "\perp"] \arrow[dr] & & \infty\mathrm{Cat}_* \arrow[ll, shift right, "\Sigma^\infty"'] \arrow[dl, "\mathrm{Free}"', "\perp"] \\
& \mathrm{CMon}(\infty\mathrm{Cat}) \simeq: \mathrm{CatSp}^{cn} \arrow[ul, bend left, "B^\infty", "\perp"'] \arrow[ur] &
\end{tikzcd}$$

R
$\mathrm{Mod}_R(\mathrm{Sp})$
$\mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^L_{\omega,1})$
$\vdots$

ΩX : mon.
$\Omega^2 X$: braided mon. $\quad S^0 \mapsto \bigsqcup B\Sigma_n = \mathrm{Fin}^{\simeq}$
$\vdots$ $\quad X \mapsto \bigsqcup_n X^{\wedge n}/\Sigma_n$

ex $\Sigma^\infty S^0 = B^\infty \mathrm{Free}_{\mathbb{E}_\infty/\mathbb{E}_0} S^0$
$= B^\infty(\mathrm{Fin}^{\simeq}) =: \mathbb{F}$
$\downarrow (-)^{gp}$
$\mathbb{S}$ "BPQ thm"

(More interesting examples : ask ~~David~~ David / Stefanich's ~~thesis~~ thesis etc.)

§2 Recall : $\otimes$ of Sp is characterized by

- distributes over colim ($\leftarrow$ assume from now on)
- $\Sigma^\infty_{\mathrm{Sp}} : \mathcal{S}_* \longrightarrow \mathrm{Sp}$ is sym. mon

(cf. $\otimes$ of Ab makes Free : Set $\longrightarrow$ Ab sym mon)

Also as a sym mon cat, $(\mathrm{Sp}, \otimes) = (\mathcal{S}_*, \wedge)[(S^1)^{-1}]$.

Thm $\exists (\mathbb{E}_1\text{-})$mon. structure on CatSp promoting

$\Sigma^\infty : (\infty\mathrm{Cat}_*, \circledast) \longrightarrow (\mathrm{CatSp}, \otimes)$ to a mon fcr.

Gray smash prod

Rem. unit $= \mathbb{F}$

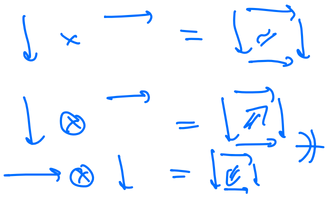
- restricts to $\mathrm{CMon}(\infty\mathrm{Cat}) \simeq \mathrm{CatSp}^{cn}$
- monoidally inverts $\vec{S}^1 = B\mathbb{N} = \overset{*}{\circlearrowleft}$

why Gray? if X : m-cat, Y : n-cat $\rightsquigarrow$ $X \times Y$: max(m,n) - cat
$X \otimes Y$: m+n - cat

X : n-cat $\rightsquigarrow$ $\Sigma X \neq \vec{S}^1 \wedge X \leftarrow$ max(n,1) - cat

(n+1) - cat $= \vec{S}^1 \circledast X$

$\hookrightarrow \infty\mathrm{Cat}_* \xrightarrow{\Sigma = \vec{S}^1 \circledast -} \infty\mathrm{Cat}_*$

ex

$\nexists$ naturally $\rightsquigarrow \mathbb{E}_1$)

but
(Can't monoidally invert a random element in a non-comm. monoid)

Key input: $\vec{S}^1$ is "central" up to the involution $(-)^\circ : \infty\mathrm{Cat}_* \longrightarrow \infty\mathrm{Cat}_*$; $\vec{S}^1 \otimes X \simeq X^\circ \otimes \vec{S}^1$

total dual

(More precisely :
$$\begin{tikzcd} \exists!\ \text{lift} \arrow[d, mapsto] & \in & HH(\infty\mathrm{Cat}_*, (-)^\circ) \arrow[d] & = & \mathrm{End}_{\mathrm{BMod}_{\infty\mathrm{Cat}_*}}(\infty\mathrm{Cat}_*, \infty\mathrm{Cat}_*^\circ) \arrow[d] \\ S^1 & \in & \infty\mathrm{Cat}_* & = & \mathrm{End}_{\mathrm{RMod}_{\infty\mathrm{Cat}_*}}(\infty\mathrm{Cat}_*) \end{tikzcd}$$
LMod str twisted by $(-)^\circ$)

§3 Stability of Sp : "fin colim are fin limits"

ex
- $\emptyset = * =: 0$
- $\sqcup = \times =: \oplus$
- fib seq = cof seq
- pb sq = po sq
- $\Sigma = \Omega^{-1}$

$\leftarrow$ true for CatSp ($\rightsquigarrow$ same property of CMon(∞Cat))

char. stability

$\leftarrow$ false even if replaced by lax analog

Some lucky shapes that happens to be both lim & colim diagram ...

Def directed / oriented pushout : Universal instance of

$$\begin{tikzcd} & Y \arrow[r] \arrow[d, "i_0"] & Z \arrow[dd] \\ Y \arrow[r, hook, "i_1"'] \arrow[d] & \Sigma^\infty_+ \square^1 \otimes Y \arrow[dr] & \\ X \arrow[rr] & & X \vec{\sqcup}_Y Z \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \arrow[dl, phantom, "\Nearrow"] \\ X \arrow[r] & X \vec{\sqcup}_Y Z \end{tikzcd}$$

More systematic def of stability?

exercise $\mathscr{C} : (\infty, 1)$-Cat. TFAE :

① $\mathscr{C}$: stable

② $\forall J$: finite cat, $\begin{tikzcd} \mathrm{Fun}(J, \mathscr{C}) \arrow[r, shift left=2] & \mathscr{C} \arrow[l, shift right=4, "\perp"'] \arrow[l, shift left=2, "\Delta", "\perp"'] \end{tikzcd}$

colim = $p_\#$?

notation $p^\#$?

③ $J = \emptyset$, 2pt, $\ulcorner$ ―――― 〃 ――――

Universal example : $\mathscr{C} = \mathrm{Sp}$ (or $\mathrm{Sp}^{\mathrm{fin}}$)

$W^L \leftarrow W\otimes - \mapsto \mathbb{S}$

$$\begin{tikzcd} \mathrm{Fun}(J, \mathrm{Sp}) \arrow[r, shift left=2] & \mathrm{Sp} \arrow[l, shift right=4, "\perp"'] \arrow[l, shift left=2, "\Delta", "\perp"'] \end{tikzcd}$$

$\underset{J}{\mathrm{colim}} = \underset{J}{\lim}{}^{W^L}$ weighted limit

"colimits are limits"

More symmetrically : make colim also weighted by $J^{op} \xrightarrow{W} \mathcal{S}_*^{\mathrm{fin}}$ fin

$\mapsto$ $\mathrm{colim}^W = \lim^{W^L}$ $\exists W^L$ "left-dual" weight

(i.e. W is an absolute weight for Sp-enrichment)

$J = \emptyset$

$\rightsquigarrow$ $\begin{tikzcd} \underset{*}{\overset{\shortparallel}{\mathrm{Fun}(\emptyset, \mathscr{C})}} \arrow[r, shift left=1, "\emptyset"] & \mathscr{C} \arrow[l, shift left=4, "!"', "\perp"] \arrow[l, shift left=-2, "\Delta = !", "\perp"'] \end{tikzcd}$ $\rightsquigarrow \emptyset = * =: 0$

$J = \overset{0}{*}\ \overset{1}{*}$

$\rightsquigarrow$ $\begin{tikzcd} \mathscr{C} \arrow[r, shift right=1] & \mathscr{C}\times\mathscr{C} \arrow[l, shift right=3, "ev_0"', "\perp"] \arrow[r, shift left=1, "\sqcup"] & \mathscr{C} \arrow[l, shift right=4, "L"', "\perp"] \arrow[l, shift left=-2, "\Delta", "\perp"'] \end{tikzcd}$

$\begin{tikzcd} c \arrow[r, mapsto] & (c, \underset{0}{\overset{\shortparallel}{1}}) \arrow[r, mapsto, "\sqcup"'] & c \\ d \arrow[u] \arrow[r, leftrightarrow] & (d, e) \arrow[u] & \end{tikzcd}$

$ev_1 \circ L = \mathrm{id}$
$\rightsquigarrow L = \Delta$
$\rightsquigarrow \sqcup = \times$.

$J = \begin{tikzcd}[sep=small] 0 \arrow[r] \arrow[d] & 1 \\ 2 & \end{tikzcd}$

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=40, "L"] \arrow[r, shift left=1, "\Delta", "\perp"'] & \mathrm{Fun}(\ulcorner, \mathscr{C}) \arrow[l, shift right=3, "\mathrm{colim}"', "\perp"] \arrow[l, shift left=3, "\lim = ev_0", "\perp"'] \arrow[d, shift right=2, "ev_1"'] \\ & \mathscr{C} \arrow[u, shift right=2, "\begin{smallmatrix} \mathrm{id} \to 0 \\ \downarrow \\ \mathrm{id} \end{smallmatrix}"'] \end{tikzcd}$$

$\mathscr{C} \xrightarrow{\begin{smallmatrix} \mathrm{id} \to 0 \\ \downarrow \\ 0 \end{smallmatrix}} \mathrm{Fun}(\ulcorner, \mathscr{C})$

$$\underbrace{\begin{tikzcd} \mathscr{C} \arrow[r, shift left=1, "L"] & \mathscr{C} \arrow[l, shift left=1, "\mathrm{colim}", "\perp"'] \arrow[r, shift left=1, "ev_0"] & \mathscr{C} \arrow[l, shift left=1, "\begin{smallmatrix} \mathrm{id} \to 0 \\ \downarrow \\ 0 \end{smallmatrix}", "\perp"'] \arrow[ll, bend left=40, "\Sigma"] \end{tikzcd}}$$

$\rightsquigarrow L = \begin{tikzcd}[sep=small] \Sigma^L \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd}$

fully faithful

$\begin{tikzcd} \mathscr{C} \arrow[r, shift left=1, "\Sigma^L"] & \mathscr{C} \arrow[l, shift left=1, "\Sigma", "\perp"'] \end{tikzcd}$

but Σ : colim
$\rightsquigarrow \Sigma^L \circ \Sigma \simeq \Sigma \circ \Sigma^L$
$\simeq \mathrm{id}$.
So Σ : invertible 〃

SW duality : $J = * \rightsquigarrow X \in \mathrm{Sp}$ dualizable $\iff X \simeq \Sigma^{\infty - n} \overset{\exists}{} Y$, $Y \in \mathcal{S}_*^{\mathrm{fin}}$

Thm dir. po are absolute in CatSp :

$$- \vec{\sqcup} - : \begin{tikzcd} \mathrm{Fun}(\ulcorner, \mathrm{CatSp}) \arrow[r, "\perp"'] & \mathrm{CatSp} \arrow[l, shift right=3] \arrow[l, shift left=3, "{\left[\begin{smallmatrix} \square^1 \hookleftarrow \{1\} \\ \uparrow \\ \{0\} \end{smallmatrix}, -\right]}", "\perp"'] \end{tikzcd}$$

$\Sigma^{-1}(I^{op} \longrightarrow \vec{S}^1 \longleftarrow I)$

Cor

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[rr] \arrow[d] \arrow[dl, phantom, "\Nearrow"] & & 0 \arrow[d] \\ 0 \arrow[r] & \vec{\mathrm{cof}}\, f \simeq \vec{\mathrm{fib}}\, \Sigma f \arrow[rr] \arrow[d] \arrow[urr, phantom, "\square"] & & \Sigma X \arrow[d, "\Sigma f"] \\ & 0 \arrow[rr] \arrow[urr, phantom, "\Nearrow"] & & \Sigma Y \end{tikzcd}$$

"Baratt-Puppe sequence"

• $\forall$ finite n-poset?

Thm TFAE for $X \in \mathrm{CatSp}$: ① X : dualizable $\qquad \Sigma^{\infty} Y \quad Y \in \infty\mathrm{Cat}^{\text{lax-fin}}$

SW duality for ∞-cats

② $X \in$ full sub gen by $\mathbb{F}$ under $0, \vec{\sqcup}$, retract, Σ^{-1}

③ $[X,-]_{(\text{op})\text{lax}} : \mathrm{CatSp} \to \mathrm{CatSp}$ comm. w/ pushouts & filt colim

§4 Cob hyp ⇒ Cob hyp w/ singularities (sketched in Lurie's TFT paper §4.3

free d-Cat w/ duals & adjoints on an obj w/ $O(d)$-action

Recall $\mathrm{Bord}_d \in \mathrm{CMon}(d\mathrm{Cat})$ - 0-mor = 0-mfd

(fully ext. unori) - 1-mor = 1-mfd. cobord b/w 0-mflds

⋮

- k-mor = k $(\leq d)$ —— ~ (k-1) —

ex

$$\begin{tikzcd}
\overset{\text{unori}}{\mathrm{Bord}_{d-1}} \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "⇗ ② \ulcorner"] & \mathrm{Bord}_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Bord}_{d,\partial} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner ① ⇗"] & B\,\mathrm{Bord}_{d-1} \arrow[d, "Bi"] \\
 & 0 \arrow[r] & B\,\mathrm{Bord}_d
\end{tikzcd}$$

$\mathrm{Bord}_{d,\partial} := \overrightarrow{\mathrm{fib}}\ Bi$

$\simeq \overrightarrow{\mathrm{cof}}\ i$

① as $\overrightarrow{\mathrm{fib}}$, have description of cells :

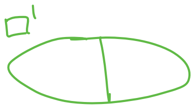
Domain wall

$\mathrm{Bord}_{d-1}^{SO(d-1)} \to \mathrm{Bord}_d^{O(d) \sqcup O(d)}$

$\to \mathbb{F}[-d]$

$\infty\mathrm{Cat} \to \mathrm{CatSp} \to \mathrm{CatSp}^{\text{dual}}$

$\mathbb{F}[d-1] \to \mathbb{F}[d] \oplus \mathbb{F}[d]$



e.g. $\vec{S}^k \to \mathrm{Bord}_{d,\partial}$

$$\begin{tikzcd}
\vec{S}^k \arrow[r] \arrow[d] \arrow[dr, phantom, "⇗"] & B\mathrm{Bord}_{d-1} \arrow[d] \\
0 \arrow[r, "\phi"'] & B\,\mathrm{Bord}_d^{2pt}
\end{tikzcd}$$

$$\begin{tikzcd}
* \arrow[r, "\partial M"] \arrow[d] \arrow[dr, "\phi"', bend right] \arrow[dr, phantom, "⇗ M"] & \Omega^{k-1}\mathrm{Bord}_{d-1} \arrow[d] \\
0 \arrow[r, "\phi"'] & \Omega^{k-1}\mathrm{Bord}_d^{2pt}
\end{tikzcd}$$

i.e. k-mfd w/ ∂ $(M \supset \partial M)$

② as $\overrightarrow{\mathrm{cof}}$, have map-out property : (fact : $\mathrm{SM}\infty\mathrm{Cat}^{\text{dual}} \subset \mathrm{SM}\infty\mathrm{Cat}$ closed under extensions

$\mathrm{Bord}_{d,\partial} \xrightarrow{Z} \mathcal{C}$

$$\begin{tikzcd}
\mathrm{Bord}_{d-1} \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "⇗ \ulcorner"] & \mathrm{Bord}_d \arrow[d, "Z_0"] \\
0 \arrow[r, "\mathbb{1}"'] & \mathcal{C}
\end{tikzcd}$$

$\mathcal{C} \in \mathrm{SM}d\mathrm{Cat}^{\text{dual}}$

obvious generalization
+
iteration
↓
Lurie's version
general

$Z_0 \overset{\text{Cob}}{\underset{\text{hyp}}{\leftrightsquigarrow}} Z_0(*)$: obj of $\mathcal{C}$ w/ $O(n)$-action

↑ ~ $O(d-1)$-equivariant

$\mathbb{1}_{\mathcal{C}}$