A lax guide to oplax tensor product conventions

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People seem to agree that there is no consensus about the conventions for lax and oplax Gray tensor products. Some even make jokes about it. However, one convention is in fact more common than the other, and there is a reasonable explanation for this.

The purpose of these mathematically inessential notes is to explain this convention clearly, so that I can stick to it for the rest of my life, and perhaps also peer-pressure a few people into following it. Even when you encounter a different convention, having your own convention firmly settled should at least reduce the amount of time spent being confused. I hope that, in this small and indirect way, these notes will contribute to mathematics by saving mathematicians' time.

Remark. Nothing here depends on univalence, so an ∞-category means a valent/flagged one.

(op)lax natural transformations

The logical order would be to start with Steiner theory, define the (op)lax Gray tensor products, and then define (op)lax natural transformations. However, the convention for natural transformations is the least ambiguous, and our main purpose is to fix convention. So we will begin here, and we will not try to give a fully precise definition.

Definition. For a pair of functors $F, G : \mathcal{C} \to \mathcal{D}$ of ∞-categories, a *lax natural transformation* $\alpha : F \to G$ consists of the following data:

- a morphism $\alpha_x : Fx \to Gx$ in $\mathcal{D}$ for every object $x \in \mathcal{C}$;
- a 2-morphism $\alpha_f : Gf \circ \alpha_x \Rightarrow \alpha_y \circ Ff$ for every morphism $f : x \to y$ in $\mathcal{C}$;
- a 3-morphism $\alpha_\theta : (\alpha_y * F\theta) \circ \alpha_f \Rrightarrow \alpha_g \circ (G\theta * \alpha_x)$ for every 2-cell $\theta : f \Rightarrow g : x \to y$ in $\mathcal{C}$;
- and so on.

$$\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\alpha_x"] & Fy \arrow[d, "\alpha_y"] & & Fx \arrow[r, bend left=50, "Fg", ""{name=A, below}] \arrow[r, bend right=50, "Ff"', ""{name=B}] \arrow[d, "\alpha_x"] & Fy \arrow[d, "\alpha_y"] & & Fx \arrow[r, bend left=50, "Fg"] \arrow[d, "\alpha_x"'] & Fy \arrow[d, "\alpha_y"] & \\
Gx \arrow[r, "Gf"'] \arrow[ur, Rightarrow, "\alpha_f"] & Gy, & & Gx \arrow[r, bend right=50, "Gf"'] \arrow[ur, Rightarrow, "\alpha_f"'] & Gy \arrow[r, phantom, "\overset{\alpha_\theta}{\Rrightarrow}"] & {} & Gx \arrow[r, bend left=50, "Gg", ""{name=C, below}] \arrow[r, bend right=50, "Gf"', ""{name=D}] \arrow[ur, Rightarrow, "\alpha_g"] & Gy, & \cdots.
\arrow[Rightarrow, from=B, to=A, "F\theta"]
\arrow[Rightarrow, from=D, to=C, "G\theta"]
\end{tikzcd}$$

The point is that for every cell $c : c_0 \to c_1$ of $\mathcal{C}$, one gets a component map α_c from (α_{c_0} + a cell) to (α_{c_1} + a cell). "A cell" is one of Fc or Gc, alternating in the dimension. An *oplax natural transformation* $\alpha : F \to G$ consists of the following data:

- a morphism $\alpha_x : Fx \to Gx$ in $\mathcal{D}$ for every object $x \in \mathcal{C}$;
- a 2-morphism $\alpha_f : \alpha_y \circ Ff \Rightarrow Gf \circ \alpha_x$ for every morphism $f : x \to y$ in $\mathcal{C}$;
- a 3-morphism $\alpha_\theta : \alpha_g \circ (\alpha_y * F\theta) \Rrightarrow (G\theta * \alpha_x) \circ \alpha_f$ for every 2-cell $\theta : f \Rightarrow g : x \to y$ in $\mathcal{C}$;
- and so on.

$$\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\alpha_x"] & Fy \arrow[d, "\alpha_y"] \arrow[dl, Rightarrow, shorten=4mm, "\alpha_f"'] \\
Gx \arrow[r, "Gf"] & Gy,
\end{tikzcd}
\qquad
\begin{tikzcd}
Fx \arrow[r, bend left, "Ff", ""{name=U, below}] \arrow[r, bend right, "Fg"', ""{name=D}] \arrow[d, "\alpha_x"] & Fy \arrow[d, "\alpha_y"] \arrow[dl, Rightarrow, shorten=4mm, "\alpha_g"] & {} \arrow[d, phantom, "\overset{\alpha_\theta}{\Rrightarrow}"] & Fx \arrow[r, bend left, "Ff"] \arrow[d, "\alpha_x"'] & Fy \arrow[d, "\alpha_y"] \arrow[dl, Rightarrow, shorten=4mm, "\alpha_f"'] & \\
Gx & Gy & {} & Gx \arrow[r, bend left, "Gf", ""{name=U2, below}] \arrow[r, bend right, "Gg"', ""{name=D2}] & Gy, & \cdots.
\arrow[Rightarrow, from=U, to=D, "F\theta"]
\arrow[Rightarrow, from=U2, to=D2, "G\theta"]
\end{tikzcd}$$

The point is that for every cell $c : c_0 \to c_1$ of $\mathcal{C}$, one gets a component map α_c from $(Fc + \alpha_{c_1})$ to $(Gc + \alpha_{c_0})$.

In other words, *lax* natural transformations are the ones whose components point in the same direction as the cells being evaluated. Oplax natural transformations, on the other hand, insist that the transformation itself should always flow from F to G, at the cost of making the components point backwards relative to the cells being evaluated, hence "op" in "oplax."

(op)lax Gray tensor products

In a nonsymmetric monoidal closed category, the default good practice for internal homs is

$$\mathrm{Hom}(X, [Y, Z]) \simeq \mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(Y, Z^X).$$

That is, we write the right internal hom as a left action, and the left internal hom as a right action.[1] This convention has the pleasant feature that composition introduces no flip of order:

$$[X, [Y, Z]] \simeq [X \otimes Y, Z], \qquad (Z^X)^Y \simeq Z^{X \otimes Y}.$$

In other words, by default we regard $[Y, -]$ as the right internal hom, meaning the right adjoint to tensoring from the right, $(-) \otimes Y$. Following this convention, we label the two Gray tensor products by the corresponding internal homs:

$$\mathrm{Map}(X \otimes^{\mathrm{lax}} Y, Z) \simeq \mathrm{Map}(X, \mathrm{Fun}^{\mathrm{lax}}(Y, Z)), \quad \mathrm{Map}(X \otimes^{\mathrm{oplax}} Y, Z) \simeq \mathrm{Map}(X, \mathrm{Fun}^{\mathrm{oplax}}(Y, Z)).$$

As will be clear in the next section, $\otimes^{\mathrm{oplax}}$ is the reverse of $\otimes^{\mathrm{lax}}$:

$$X \otimes^{\mathrm{oplax}} Y = Y \otimes^{\mathrm{lax}} X.$$

Together with the convention for natural transformations above, this determines which tensor product deserves to be called lax and which one deserves to be called oplax. This part should not be ambiguous.

What is ambiguous, and genuinely a matter of convention, is whether we set $\otimes = \otimes^{\mathrm{lax}}$ or $\otimes = \otimes^{\mathrm{oplax}}$. The convention $\otimes = \otimes^{\mathrm{oplax}}$ is probably the majority convention. It also fits rather nicely with our left-to-right writing system.[2]

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[1] By the way, this forces the composition convention $\circ : [Y, Z] \otimes [X, Y] \to [X, Z]$, which is probably more authentic, though not my favorite. Nevertheless, I choose to accept it for consistency.

[2] In my thesis I used $\otimes = \otimes^{\mathrm{lax}}$ for a reason. As I write these notes, I am beginning to regret it.

In terms of Steiner theory

Steiner theory allows us to compare strict ∞-categories with chain complexes, and gives the most convenient way to define the (op)lax Gray tensor products.

Theorem. (1) *There is an adjunction*

$$\lambda : \{\textit{strict } \infty\textit{-categories}\} \rightleftarrows \{\textit{augmented directed chain complexes}\} : \nu.$$

This restricts to an equivalence on 'Steiner' objects, namely the free and loop-free ones. These include the usual combinatorial building blocks of strict ∞-categories, such as Θ, fully lax cubes, and orientals.

(2) *The* ***oplax*** *Gray tensor product is the unique biclosed monoidal structure on strict ∞-categories making λ monoidal, and making the restriction of ν to Steiner complexes monoidal, with respect to the tensor product of augmented chain complexes.*[3] *Here the tensor product of chain complexes uses the Koszul sign rule*

$$\partial(x \otimes y) = (\partial x) \otimes y + (-1)^{\deg(x)} x \otimes (\partial y).$$

The ***lax*** *Gray tensor product corresponds instead to the* reverse *Koszul sign rule*

$$\partial(x \bar{\otimes} y) = (-1)^{\deg(y)} (\partial x) \bar{\otimes} y + x \bar{\otimes} (\partial y).$$

The Koszul sign rule makes it quite visible where the 'op' in oplax natural transformations comes from. Recall that an oplax natural transformation $\alpha : F \to G : \mathcal{C} \to \mathcal{D}$ is the same thing as a map

$$[1] \to \mathrm{Fun}^{\mathrm{oplax}}(\mathcal{C}, \mathcal{D}),$$

or equivalently a map

$$[1] \otimes^{\mathrm{oplax}} \mathcal{C} \to \mathcal{D}.$$

For a cell $c : \mathbb{D}^n \to \mathcal{C}$, the component α_c is given by the top cell of $[1] \otimes \mathbb{D}^n$. The chain complexes corresponding to $[1]$ and $\mathbb{D}^n$ are the cellular chain complexes for the usual CW structures. Let x and y denote the top cells of $[1]$ and $\mathbb{D}^n$, respectively. Then the Koszul sign rule gives

$$\partial(x \otimes y) = (\underline{1} - \underline{0}) \otimes y - x \otimes \partial y,$$

because $\deg(x) = 1$. Here $\underline{0}$ and $\underline{1}$ denote the 0-cells represented by the vertices $0, 1 \in [1]$. Negative and positive parts of the boundary correspond to cells in the domain and codomain, respectively. Thus the first term $(\underline{1} - \underline{0}) \otimes y$ says that the top of the cylinder (Fc) lies in the domain and the bottom (Gc) lies in the codomain. The second term

$$-x \otimes \partial y = x \otimes \partial^- y - x \otimes \partial^+ y$$

says that, on the side of the cylinder, the cylinder over the domain of y appears in the codomain, while the cylinder over the codomain of y appears in the domain. In other words, the direction on the side gets flipped.

If we use the reverse Koszul sign rule instead, we get

$$\partial(x \bar{\otimes} y) = (-1)^{\deg(y)} (\underline{1} - \underline{0}) \bar{\otimes} y + x \bar{\otimes} \partial y.$$

Then the domain/codomain direction on the side of the cylinder agrees with that of y, but the assignment of the top and bottom of the cylinder to the domain and codomain alternates with the dimension.

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[3] Even though the tensor product of chain complexes is symmetric monoidal, the positivity structure is not preserved by the symmetry isomorphism, so this monoidal structure is asymmetric.

The lax cube

The definition of the cube $\square^n = [1]^{\otimes n}$, whose corresponding cell complex is $(\mathbb{Z} \to \mathbb{Z}^2)^{\otimes n}$, does not depend on whether we use the lax or oplax tensor product. In other words, there is a unique isomorphism

$$[1]^{\otimes^{\mathrm{oplax}} n} = [1]^{\otimes^{\mathrm{lax}} n}$$

obtained by reversing the order of the factors. The uniqueness comes from the fact that $\mathrm{Aut}(\square^n) = \{\mathrm{id}\}$.

$$\text{oplax:}\quad 0 \longrightarrow 1,\qquad \begin{tikzcd} 00 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & 01 \arrow[d] \\ 10 \arrow[r] & 11, \end{tikzcd}\qquad \begin{tikzcd}[column sep=small]
 & & 001 & & & & & & 001 & & \\
000 & & \Downarrow & & 011 & & 000 & & & & 011 \\
 & & 010 & & & \Rrightarrow & & \Swarrow & 101 & \Swarrow & \\
100 & \Swarrow & & \Swarrow & 111 & & 100 & & \Downarrow & & 111. \\
 & & 110 & & & & & & 110 & &
\arrow[from=2-1, to=1-3] \arrow[from=1-3, to=2-5] \arrow[from=2-1, to=3-3] \arrow[from=3-3, to=2-5] \arrow[from=2-1, to=4-1] \arrow[from=2-5, to=4-5] \arrow[from=3-3, to=5-3] \arrow[from=4-1, to=5-3] \arrow[from=5-3, to=4-5]
\arrow[from=2-7, to=1-9] \arrow[from=1-9, to=2-11] \arrow[from=2-7, to=4-7] \arrow[from=1-9, to=3-9] \arrow[from=2-11, to=4-11] \arrow[from=4-7, to=3-9] \arrow[from=3-9, to=4-11] \arrow[from=4-7, to=5-9] \arrow[from=5-9, to=4-11]
\end{tikzcd}$$

$$\text{lax:}\quad 0 \longrightarrow 1,\qquad \begin{tikzcd} 00 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & 01 \arrow[d] \\ 10 \arrow[r] & 11, \end{tikzcd}\qquad \begin{tikzcd}[column sep=small]
 & & 001 & & & & & & 001 & & \\
000 & & \Uparrow & & 011 & & 000 & & & & 011 \\
 & & 010 & & & \Rrightarrow & & \Nearrow & 101 & \Nearrow & \\
100 & \Nearrow & & \Nearrow & 111 & & 100 & & \Uparrow & & 111. \\
 & & 110 & & & & & & 110 & &
\arrow[from=2-1, to=1-3] \arrow[from=1-3, to=2-5] \arrow[from=2-1, to=3-3] \arrow[from=3-3, to=2-5] \arrow[from=2-1, to=4-1] \arrow[from=2-5, to=4-5] \arrow[from=3-3, to=5-3] \arrow[from=4-1, to=5-3] \arrow[from=5-3, to=4-5]
\arrow[from=2-7, to=1-9] \arrow[from=1-9, to=2-11] \arrow[from=2-7, to=4-7] \arrow[from=1-9, to=3-9] \arrow[from=2-11, to=4-11] \arrow[from=4-7, to=3-9] \arrow[from=3-9, to=4-11] \arrow[from=4-7, to=5-9] \arrow[from=5-9, to=4-11]
\end{tikzcd}$$

In both cases, the domain and codomain are glued around the central vertices of the form

$$0101\cdots \qquad \text{and} \qquad 1010\cdots\,.$$

In oplax notation, $0101\cdots$ is always in the domain and $1010\cdots$ is always in the codomain (we look at the first digit), whereas in lax notation we look at the last digit instead.

More generally, when deciding the direction of a cell in an oplax-labeled cube, we read the binary string of the center point from left to right and apply the rule "small to large." For a lax-labeled cube, we instead read the binary string from right to left and apply the same rule. For instance, in the face containing $001, 101, 011, 111$, the inequality

$$011 < 101$$

says that, in the oplax cube, the 2-morphism goes from the composite containing 011 to the composite containing 101.

The "odd-to-even" rule

A useful rule of thumb for coherently assigning directions to cells in a family of 'polytopal' ∞-categories is that cells should go from odd faces to even faces. This is basically the Koszul sign rule in disguise, but the convention was especially useful in the pre-Steiner age of parity complexes.

For orientals, the meaning of this rule should be clear: the composite of the odd faces is the domain, and the composite of the even faces is the codomain. Here one should remember that, in $[1] = (0 \to 1)$, the vertex 0 is the 1st face, while the vertex 1 is the 0th face. The point is that, if we stick to this rule in all dimensions, the odd faces and even faces really do compose to single cells.

For n-cubes, label the facets by

$$\partial_{i,\epsilon} \qquad (1 \leq i \leq n,\ \epsilon \in \{0,1\}),$$

where $\partial_{i,\epsilon}$ means "the ith coordinate is ϵ." In the oplax cube, we declare the parity of $\partial_{i,\epsilon}$ to be the parity of $i + \epsilon$. I learned this convention from the comical sets paper. For the lax cube, one must read the binary strings from right to left, so the parity of $\partial_{i,\epsilon}$ is instead the parity of

$$(n + 1 - i) + \epsilon.$$

For instance, the 3-cube is labeled as follows, and one can check that the odd-to-even rule is obeyed:

$$\begin{tikzcd}
{} & 001 \arrow[dr] & {} & {} & {} & 001 \arrow[dr] \arrow[dd] & {} \\
000 \arrow[ur] \arrow[dr] \arrow[dd] & \partial_{1,0} & 011 \arrow[dd] & {} & 000 \arrow[ur] \arrow[dd] & {} & 011 \arrow[dd] \\
\partial_{3,0} & 010 \arrow[ur] \arrow[dd] & \partial_{2,1} & \Rrightarrow & \partial_{2,0} & 101 \arrow[dr] & \partial_{3,1} \\
100 \arrow[dr] & {} & 111 & {} & 100 \arrow[ur] \arrow[dr] & \partial_{1,1} & 111. \\
{} & 110 \arrow[ur] & {} & {} & {} & 110 \arrow[ur] & {}
\end{tikzcd}$$

Hom objects via tensor products

Returning to the natural-transformation side, we find another reason why the oplax tensor product may be the better default. Namely, if we want the following formula to appear without opposites, then we must use $\mathrm{Fun}^{\mathrm{oplax}}$ rather than $\mathrm{Fun}^{\mathrm{lax}}$.

Proposition. *Let X be an ∞-category, and let*

$$(s_{k-1}, t_{k-1}) : \partial C_k \to X$$

be a pair of parallel $(k-1)$-morphisms. Then there is a pullback square

$$\begin{tikzcd}
X(s_{k-1}, t_{k-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{\mathrm{oplax}}(C_k, X) \arrow[d] \\
* \arrow[r, "{(s_{k-1}, t_{k-1})}"] & \mathrm{Fun}^{\mathrm{oplax}}(\partial C_k, X).
\end{tikzcd}$$