0. Review

Gro topos : lex localization of a presheaf cat

$$\mathcal{E} \underset{\text{f.f.}}{\overset{\text{lex}}{\rightleftarrows}} \mathrm{PSh}(\mathcal{C})$$

equivalently $\cong \mathrm{Sh}(\mathcal{C}, \tau)$ $\exists \tau$: Gro topology.

Satisfy
- (i) $\exists$ finite lim
- (ii) cart closed
- (iii) $\exists$ subobj classifier

① **Def** $\mathcal{E}$ elementary topos if (i) – (iii).

② $f : \mathcal{E} \to \mathcal{F}$ Geom mor is

$$\mathcal{E} \underset{f_*}{\overset{f^* : \text{lex}}{\rightleftarrows}} \mathcal{F} \quad (\perp)$$

$\rightsquigarrow$ Topos : (2-) cat of (elementary) topoi

i.e. $\mathrm{Topos}(\mathcal{E}, \mathcal{F}) \simeq \mathrm{Fun}^{\text{lex ladj}}(\mathcal{F}, \mathcal{E})$

I. Examples

ex 1 $\mathrm{Top} \to \mathrm{Loc} \overset{\text{f.f.}}{\hookrightarrow} \mathrm{Topos}$

$\mathrm{Sob} \subset \mathrm{Top}$, $\mathrm{SpLoc} \subset \mathrm{Loc}$, $\mathrm{Sob} \overset{\sim}{\to} \mathrm{SpLoc}$

in particular $f : X \to Y \in \mathrm{Top}$

$$\rightsquigarrow \mathrm{Sh}(X) \underset{f_* = (-)\circ f^{-1}}{\overset{f^*}{\rightleftarrows}} \mathrm{Sh}(Y) \quad (\perp)$$

(Ét : top sp & loc homeo)

$$\text{Ét}_{/X} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \text{Ét}_{/Y} \quad (\perp)$$

f^* $\leftarrow$ pullback

Def geom mor is essential if f^* has ladj $f_!$

$$f_! \dashv f^* \dashv f_*$$

ex 2 Special case : $f : X \to Y$ is étale $\in \text{Ét}_{/Y}$

Then $\text{Ét}_{/X} \cong (\text{Ét}_{/Y})_{/X} \underset{f_*}{\overset{f_!,\ f^*}{\leftrightarrows}} \text{Ét}_{/Y} \ni X$

$\downarrow$

$\mathrm{Sh}(X) \simeq \boxed{\mathrm{Sh}(Y)_{/F_X} \overset{f_! = (-)\circ f}{\leftrightarrows} \mathrm{Sh}(Y)} \ni F_X$

Def $\mathcal{E} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{F}$ étale if equiv. to

$$\mathcal{F}_{/X} \underset{\pi_*}{\overset{\pi^*}{\leftrightarrows}} \mathcal{F} \qquad X \xrightarrow{\pi} *$$

generalizing

ex. 3 **Fact** ① $\mathcal{E}$: topos $\ni X$ $\rightsquigarrow$ $\mathcal{E}_{/X}$: topos

② Any $f : X \to Y$ induce an ess geom mor

$$\mathcal{E}_{/X} \underset{f_* = \Pi_f}{\overset{f_! = \Sigma_f,\ f^*}{\leftrightarrows}} \mathcal{E}_{/Y}$$

étale $\rightsquigarrow$ $(\mathcal{E}_{/Y})_{/X}$

where
- f^* : pullback
- $f_! = f \circ (-)$
- $\Pi_f = f_*$: "objects of sections" or "$(\Pi_f E)_y = \prod_{f(x)=y} E_x$"

exer slice Gro top is Gro top.

Hint First prove $\mathrm{PSh}(\mathcal{C})_{/F} \simeq \mathrm{PSh}(\int F)$ $\leftarrow$ cat of elements

obs If f : epi, then f^* : faithful (exer).
If f : mono, then f_* : fully faithful.

Def $\mathcal{E} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{F}$ is an embedding if f_* : f.f. (i.e. lex loc)

More Generally: Thm Factorization (cf) epi-mono factorization : Unique Surjection - f. f^* : faithful

ex. 4 $G \subset \mathcal{E}$ lex comonad. $\overset{\text{thm}}{\rightsquigarrow} \mathrm{coAlg}_G(\mathcal{E})$: topos

$(-)\times X$

$$\rightsquigarrow p : \mathcal{E} \underset{p_* : \text{Cofree}}{\overset{\pi_! = p^* \text{ : forget : lex by } G : \text{lex}}{\leftrightarrows}} \mathrm{coAlg}_G(\mathcal{E}) \quad : \text{surj. Geom mor.}$$

$\pi^* =$ "$\mathcal{E}_{/X}$"

Thm $G = f_* f^*$

$\mathcal{E} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{F}$ surj ($\Longleftrightarrow$ comonadic)

$\mathcal{F} \cong \mathrm{coAlg}(\mathcal{E})$, $P : \mathcal{E} \to \mathrm{coAlg}(\mathcal{E})$

proof Beck's Thm + $\mathcal{E}$.

ex. 5 $\mathcal{E}$: topos, $1 \xrightarrow{t} \Omega$

Def LT topology on $\mathcal{E}$ is a morph $j : \Omega \to \Omega$ s.t. $j^2 = j$, $jt = t$, $j(- \wedge -) = j(-) \wedge j(-)$

$\rightsquigarrow \mathrm{Sub}(E) \simeq \mathcal{E}(E, \Omega)$ "Kernel"

$\circlearrowleft A \mapsto \bar{A}$ closure operator $\qquad \circlearrowleft j$

Def
- $A \rightarrowtail E$ dense if $\bar{A} = E$ (if $j \circ \chi_A = \chi_E$.)
- $F \in \mathcal{E}$: j-sheaf if $A \to F$, $\forall$ dense $A \rightarrowtail E$, $\exists! \, E \to F$

$\rightsquigarrow \mathrm{Sh}_j(\mathcal{E}) \underset{}{\overset{a}{\leftrightarrows}} \mathcal{E} \quad (\perp)$

Fact
- $\exists a$: lex "sheafification"
- $\forall$ emb ($\mathcal{F} \leftrightarrows \mathcal{E}$) $\exists j$ on $\mathcal{E}$ $\mathcal{F} \simeq \mathrm{Sh}_j(\mathcal{E})$
- For $\mathcal{E} = \mathrm{PSh}(\mathcal{C})$
  LT top $\longleftrightarrow$ Gro top on $\mathcal{C}$
  by "$S \rightarrowtail よ c$ is dense iff covering".
  + codensity (later)

example 6. top sp $\overset{\text{pts}}{*} \xrightarrow{x} X$

$$\rightsquigarrow \quad \mathrm{Set} \underset{x_* = \mathrm{sky}_x}{\overset{x^* = \mathrm{stalk}_x}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{Sh}(X)$$

Def • A pt of $\mathcal{E}$ is a geom mor $\mathrm{Set} \rightleftarrows \mathcal{E}$.

• $\mathcal{E}$ has enough pts if stalks jointly detects isom.

Want to understand pts of Gro topos $\mathrm{Sh}(\mathcal{C}, \tau)$ using this presentation. (or (geom into))

II A Ubiquitous Construction (cat hmtpy thry chap 1)

Def
$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\Downarrow\theta"] & \mathcal{E} \\ \mathcal{D} \arrow[ur, "K_!F"'] \arrow[ur, bend right, "\forall G"'] \end{tikzcd}$$
left Kan ext of F along K is a pair $\begin{cases} K_!F : \mathcal{D} \to \mathcal{E} \\ \theta : F \to K_!F \circ K \end{cases}$ $\exists!$ $\forall\eta$ $\forall G$

Universal among such

$$\Longleftrightarrow \mathcal{E}^{\mathcal{D}}(K_!F, G) \simeq \mathcal{E}^{\mathcal{C}}(F, G \circ K)$$
$$G \circ K = K^*G$$

exercise If K : fully faithful,
$\theta : F \Rightarrow K_!F \circ K$ isom

Pointwise Kan ext if $\mathcal{E}$: cocomp, $\mathcal{C}$ small, $K_!F$ always exists by

$$\begin{tikzcd} \mathrm{colim}\,[K\downarrow d \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \mathcal{C} \arrow[r, "F"] \arrow[d, "K"] \arrow[dr, phantom, "\Downarrow"] & \mathcal{E}] =: K_!F(d) \\ \mathbb{1} \arrow[r, "d"'] & \mathcal{D} \arrow[ur, "K_!F"'] & \end{tikzcd}$$

ex "$\otimes$ - Hom" adjunction "left $\mathcal{C}$-module"

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\sharp"'] & \mathcal{E} : \text{cocomplete} \\ \mathrm{Psh}(\mathcal{C}) \arrow[ur, shift left, "- \otimes_{\mathcal{C}} F = \sharp_! F"] \arrow[ur, phantom, "\perp"] & \arrow[l, "F^! \sharp \; \mathrm{Hom}_{\mathcal{E}}(F, -)"] \end{tikzcd}$$

$G : \mathcal{C}^{op} \to \mathrm{Set} \in$ "right $\mathcal{C}$-mod"

when $\mathcal{E} = \mathrm{Set}$ [make sense for $\forall\mathcal{E}$ since $\mathcal{E}$: "tensored"/sets]

• $\sharp_!F(G) \cong \underset{\sharp c \to G}{\mathrm{colim}}\, Fc \cong \coprod_c Gc \times Fc \,/\, ((x, \alpha_* y) \sim (\alpha^* x, y))$

$x \in G(c) \Leftrightarrow \left(\sharp c \xrightarrow{x} G \right)$, $\sharp c \in \sharp\downarrow G$

c, $y \in Fc$, Gc; $\alpha\downarrow$, $\downarrow\alpha_*$, $\uparrow\alpha^*$; c', Fc', $G c' \ni x$

$= \boxed{G \otimes_{\mathcal{C}} F}$

Rem if instead
$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "G"] \arrow[d, "\sharp"'] \arrow[dr, phantom, "\Downarrow"] & \mathrm{Set} \\ F \in \mathrm{Set}^{\mathcal{C}} \arrow[ur] \arrow[ur, bend right, "F \otimes_{\mathcal{C}^{op}} G"'] \end{tikzcd}$$

• $F^! \sharp(e) : c \mapsto \mathrm{Hom}_{\mathcal{E}}(Fc, e)$. $=: \boxed{\mathrm{Hom}_{\mathcal{E}}(F, -)}$
$\mathrm{Psh}(\mathcal{C})$

$- \otimes_{\mathcal{C}} F \dashv \mathrm{Hom}_{\mathcal{E}}(F, -)$

$$\mathrm{Hom}_{\mathcal{E}}(G \otimes_{\mathcal{C}} F, e) \cong \{ Gc \times Fc \xrightarrow{\alpha_c} e \}_{c \in \mathcal{C}} : \text{natural}$$
$$\updownarrow \simeq$$
$$\mathrm{Hom}_{\mathrm{Psh}(\mathcal{C})}(G, \mathrm{Hom}_{\mathcal{E}}(F, e)) \simeq \{ Gc \to \mathrm{Hom}_{\mathcal{E}}(Fc, e) \}_{c \in \mathcal{C}} : \text{natural}$$

$\rightsquigarrow$ Thm $\left.\begin{matrix} \mathcal{C} \text{ small} \\ \mathcal{E} \text{ cocomp} \end{matrix}\right]$ $\mathrm{Fun}(\mathcal{C}, \mathcal{E}) \underset{\sharp^*}{\overset{\sharp_!}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{ladj}}(\mathrm{Psh}(\mathcal{C}), \mathcal{E})$ equiv

☺ $\Rightarrow \simeq$ id by $\sharp$: fully faithful.

$\Leftarrow \simeq$ id by
$$\begin{tikzcd} & \arrow[r, "L"] & \\ \arrow[u, "\sharp"] \arrow[ur, phantom, "\Downarrow"] \arrow[urr] & & \end{tikzcd} = \begin{tikzcd} \arrow[r, "\sharp"] & \arrow[r, "L"] & \\ \arrow[u, "\sharp"] \arrow[ur, "\mathrm{id}"] \arrow[urr, "L"'] & & \end{tikzcd}$$

when $\mathcal{E} = \mathrm{Psh}(\mathcal{C})$. (L pres. ptwise Kan ext).

$\sharp_! \sharp c \cong \mathrm{Hom}_{\mathrm{Psh}(\mathcal{C})}(\sharp c, -) = \mathrm{ev}_c \rightsquigarrow \sharp_! \sharp = \mathrm{id}$

formula $\rightsquigarrow$ $P \cong (\sharp_! \sharp)(P) \simeq \underset{(\sharp c \to P) \in \sharp\downarrow P}{\mathrm{colim}} \sharp c$ "codensity"

III $\mathcal{E} \underset{\perp}{\rightleftarrows} \mathrm{Psh}(\mathcal{C})$.

Def When $- \otimes_{\mathcal{C}} F$: lex we call F flat

Thm $\mathrm{Fun}^{\mathrm{flat}}(\mathcal{C}, \mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{ladj, lex}}(\mathrm{Psh}(\mathcal{C}), \mathcal{E}) \simeq \mathrm{Topos}(\mathcal{E}, \mathrm{Psh}(\mathcal{C}))$

$F \longmapsto - \otimes_{\mathcal{C}} F$

we cheated ☹ flatness in terms of
want to characterize F itself.

Def • $(\int F)^{op}$: cat of elts $\begin{cases} \text{obj} : (c, a) \text{ s.t. } a \in Fc \leftrightarrow \sharp c \to F \\ \text{mor } (c, a) \xrightarrow{f} (c', a') \\ f : c \to c' \text{ s.t. } a \xmapsto{Ff} a' \\ \sharp c \leftarrow \sharp c' \end{cases}$

$$\begin{tikzcd} (\int F)^{op} \arrow[r, "\mathrm{pr}"] \arrow[d, "\sharp\downarrow F" description] \arrow[dr, phantom, "\Swarrow"] & \mathcal{C}^{op} \arrow[d, "\sharp"] \\ F \arrow[r] & \mathrm{Set}^{\mathcal{C}} \end{tikzcd}$$

• A small cat $\mathcal{A}$ is filtered
$\Longleftrightarrow \forall \mathcal{D} \to \mathcal{A}$ finite diagram ($\mathcal{D}$ can be $\emptyset$)
$\exists$ cocone on it

Thm For $\mathcal{C}$: finite comp. $\mathcal{C} \xrightarrow{F} \mathrm{Set}$. TFAE
① F : flat
② F : lex
③ $(\int F)^{op}$: filtered.

proof ① ⇒ ② $\mathcal{C} \xrightarrow{F} \mathrm{Set}$: lex
lim pres $\Rightarrow$ $\sharp$ $\searrow$ $\mathrm{Psh}(\mathcal{C})$ $\nearrow - \otimes F$: lex

② ⇒ ③ fin $\mathcal{D} \longrightarrow (\int F)^{op} \xrightarrow{\mathrm{pr}} \mathcal{C}^{op}$

$(C_d, a_d \in F C_d) \longmapsto C_d \longrightarrow \mathrm{colim}\, C_d =: C$
$\downarrow$ cocone $\qquad F(\cdot) \quad F(\cdot)$
$(C, a) \qquad a_d \longmapsto a$

③ ⇒ ① $\quad \overset{\text{fin}}{\mathcal{D}} \longrightarrow Set^{\mathcal{C}^{op}}$, $\quad d \mapsto G_d$

want $(\lim_d G_d) \otimes_{\mathcal{C}} F \simeq \lim_d (G_d \otimes_{\mathcal{C}} F)$

$\quad\quad\ \ F \otimes_{\mathcal{C}^{op}} \lim_d G_d \qquad \lim (F \otimes_{\mathcal{C}^{op}} G_d)$

$$\begin{tikzcd} (\int F)^{op} \arrow[r] \arrow[d] & \mathcal{C}^{op} \arrow[r, "G"] \arrow[d, "\int"] & Set \\ 1 \arrow[r, "F"'] & Set \arrow[ur, "- \otimes_{\mathcal{C}^{op}} G"'] & \end{tikzcd}$$

colim $[(\int F)^{op} \to \mathcal{C}^{op} \xrightarrow{G} Set]$ $\cong \operatorname{colim}_{(\int F)^{op} \ni (c,a)} \lim_{\mathcal{D}} G_d(c)$ $\cong \lim_{\mathcal{D}} \operatorname{colim}_{(\int F)^{op}} G_d(c)$ ☐

Rmk If finite completeness is dropped, ① ⇔ ③ ⇒ ② ($\not\Leftarrow$)

② Set can be replaced by ∀cocomp topos $\mathcal{E}$ ("filtered" interpreted internally to $\mathcal{E}$.)

IV $\quad \underset{\text{cocomp}}{\mathcal{E}} \longrightarrow \underset{\text{site}}{Sh(\mathcal{C}, \tau)}$

Q. when $\mathcal{E} \underset{f_*}{\overset{f^*}{\rightleftarrows}} PSh(\mathcal{C})$ factor through $\mathcal{E} \underset{g_*}{\overset{g^*}{\rightleftarrows}} Sh(\mathcal{C},\tau)$, $i^* \dashv \int i_*$?

More generally

Lem $\mathcal{F}$: topos w/ LT topology $\Omega \xrightarrow{j} \Omega$

⇝ TFAE for $f: \mathcal{E} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{F}$, $\quad i^* \dashv \int i_*$, $Sh_j(\mathcal{F})$

① f^* factor through i

② $\forall E \in \mathcal{E}$ $f_* E$: j-sheaf

③ $\forall$ dense mono $A \overset{u}{\rightarrowtail} X$ in $\mathcal{F}$, $f^* u$: iso. (enough to see epi)

proof ① ⇒ ② ✓

② ⇒ ① $\quad i_* i^* f_* E \simeq f_* E$, $\quad g^* := f^* i_*$ $\quad g_* :=$ lax.

② ⇔ ③

$$\begin{array}{ccc} \mathcal{E}(f^*A, E) & \simeq & \mathcal{F}(A, f_*E) \\ \uparrow f^*u & & \uparrow u \\ \mathcal{E}(f^*X, E) & \simeq & \mathcal{F}(X, f_*E) \end{array}$$

③ ⇔ $\forall u \forall E$ $\cong$ $\uparrow f^*u$ $\overset{\forall u \forall E}{\Longleftrightarrow}$ $\uparrow u$ ⇔ ② ☐

Observation When $\mathcal{F} = PSh(\mathcal{C})$ (Gro topology) enough to check for covering sieves $S \rightarrowtail よc$

because $\forall f^*A \rightarrowtail f^*X$ ($\cong$)

$$\begin{array}{ccc} f^*A & \rightarrowtail & f^*X \\ \uparrow \text{colim} & & \uparrow \text{colim} \\ f^*S_i & \xrightarrow{\cong} & f^*よc_i \end{array}$$

by codensity, dense mono

☺ $よc_i \to X \overset{\chi_{S_i}}{\underset{\chi_A}{\rightrightarrows}} \Omega \xrightarrow{j} \Omega$ $\quad$ ($よc_i \to X$, χ_X)

For any sieve $f^*S \overset{u}{\rightarrowtail} f^*よc$

$\cong$ $\operatorname{colim}_{p: よd \to S,\ (よd \downarrow S)} f^*よd$ $\quad$ $f^*よp$ $\quad$ $p \in S(d) \subset Hom(d,c)$

This cocone is $\{f^*よd \xrightarrow{f^*よp} f^*よc\}_{p \in S}$

Cor For $F: \mathcal{C} \to \mathcal{E}$.

$\mathcal{E} \underset{f_* = Hom(F,-)}{\overset{f^* = -\otimes F}{\rightleftarrows}} PSh(\mathcal{C}) \hookleftarrow Sh(\mathcal{C},\tau)$ $\quad$ Def. F: Continuous

TFAE

① f factors through $Sh(\mathcal{C},\tau)$

② cocone $\{F よd \xrightarrow{F よp} F よc\}_{p \in S}$ is a colim diagram in $\mathcal{E}$ (cov sieve $\forall S \rightarrowtail よc$,)

③ $\{Fd \xrightarrow{Fp} Fc\}_{p \in S}$ is jef.

Pf

① ⇔ ② $\operatorname{colim} f^*よD \simeq f^* \operatorname{colim} よD$ (S); $f^*よc \leftarrow f^*u$. by Lem ③.

② ⇒ ③ ✓

③ ⇒ ① enough to see

$$\begin{array}{ccc} f^*S & \longrightarrow & f^*よc \quad \text{epi} \\ \uparrow & \nearrow \text{(jef)} & \\ \coprod Fd & & \end{array}$$ ✓ ☐

upshot $\quad Topos(\mathcal{E}, Sh(\mathcal{C},\tau)) \simeq Fun^{\text{conti, flat}}((\mathcal{C},\tau), \mathcal{E})$.

V. Mor of sites

Assume $(\mathcal{C}, \tau_{\mathcal{C}})$, $(\mathcal{D}, \tau_{\mathcal{D}})$ sites, finite complete.

Def $\phi: \mathcal{C} \to \mathcal{D}$ functor is a mor of site if

- lex
- pres covers. i.e. $\forall$ cov sieve $S = \{c' \to c\}$ of c, $\phi S = \{\phi c' \to \phi c\}$ generates cov. sieve of ϕc.

e.g. $X \xrightarrow{f} Y$ top sp.

⇝ $\Omega_Y \xrightarrow{f^{-1}(-)} \Omega_X$ mor of sites

Consider $A_\phi: \mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{よ} PSh\mathcal{D} \xrightarrow{a} Sh(\mathcal{D}, \tau_{\mathcal{D}})$: lex (⇔ flat).

$$\begin{tikzcd} \mathcal{C} \arrow[d, "よ"] & \\ PSh(\mathcal{C}) \arrow[r, shift left, "-\otimes A_\phi"] \arrow[r, phantom, "\perp"] & Sh(\mathcal{D},\tau_{\mathcal{D}}) \arrow[l, shift left, "Hom(A_\phi, -) = (-)\circ\phi"] \\ Sh(\mathcal{C}, \tau_{\mathcal{C}}) \arrow[u, hook] & \end{tikzcd}$$

For $F \in Sh(\mathcal{D}, \tau_{\mathcal{D}})$

$\underset{(-)\circ\phi}{Hom(A_\phi, -)}: c \mapsto Hom_{Sh}(aよ\phi(c), F) \simeq Hom_{PSh}(よ\phi(c), F) \simeq F\phi(c)$

A_ϕ: conti by fact $\{d_i \to d\}$ generates cov sieve of d iff $\{aよd_i \to aよd\}$: jef (SGL III.7.?)

Thm $\phi: \mathcal{C} \to \mathcal{D}$ induce geom mor

$f_\phi: Sh(\mathcal{D}, \tau_{\mathcal{D}}) \underset{(-)\circ\phi}{\overset{-\otimes A_\phi}{\rightleftarrows}} Sh(\mathcal{C}, \tau_{\mathcal{C}})$