

In Examples

$$\begin{array}{ccc} \text{ex1} & \text{Top} & \xrightarrow{\text{loc}} \text{Loc} \xrightarrow{\text{fit.}} \text{Topos} \\ & \cup & \cup \\ & \text{Sub} & \xrightarrow{\sim} \text{SpLoc} \end{array}$$

in particular  $f: X \rightarrow Y \in \text{Top}$

$$\rightsquigarrow \text{Sh}(X) \xrightleftharpoons[\text{f}_*]{\text{f}^*} \text{Sh}(Y)$$

$$\begin{array}{ccc} \text{(Et: top sp)} & \text{Is} & \text{Is} \\ \text{loc homeo} & \text{Et}_X & \xleftarrow[\text{f}_*]{\text{f}^*} \text{Et}_Y \end{array}$$

Def geom morph essential if  $f^*$  has ladj  $f_!$   
 $f_! \dashv f^* \dashv f_*$

ex2 Special case:  $f: X \rightarrow Y$  is étale

$$\begin{array}{ccc} \text{Then } \text{Et}_X \cong (\text{Et}_Y)_X & \xrightleftharpoons[\text{f}_*]{\text{f}^*} & \text{Et}_Y \ni X \\ \text{Sh}(X) \cong \text{Sh}(Y)_{F_X} & \xrightleftharpoons[\pi_*]{\pi^*} & \text{Sh}(Y) \ni F_X \end{array}$$

Def  $\mathcal{E} \xrightleftharpoons[\text{f}_*]{\text{f}^*} \mathcal{F}$  étale if equiv. to  $\mathcal{F}_{F_X} \xrightleftharpoons[\pi_*]{\pi^*} \mathcal{F} \quad X \xrightarrow{\pi} F_X$

0. Review

Gro topos: lex localization of a presheaf cat

$$\mathcal{E} \xrightleftharpoons[\text{f.f.}]{\text{lex}} \text{PSh}(\mathcal{E})$$

equivalently  $\text{Sh}(\mathcal{E}, \tau) \quad \exists \tau: \text{Gro topology.}$

- satisfy
- (i)  $\exists$  finite lim
  - (ii) cart closed
  - (iii)  $\exists$  subobj classifier

Def elementary topos if (i) - (iii).

(2)  $f: \mathcal{E} \rightarrow \mathcal{F}$  Geom morph is  $\mathcal{E} \xrightleftharpoons[\text{f}_*]{\text{f}^* \cdot \text{lex}} \mathcal{F}$

$\rightsquigarrow$  Topos: (2-) out of topos (elementary)

i.e.  $\text{Topos}(\mathcal{E}, \mathcal{F}) \cong \text{Fun}^{\text{lex ladj}}(\mathcal{F}, \mathcal{E})$

generalizing

ex.3 Fact ①  $\mathcal{E}: \text{topos} \ni X \rightarrow \mathcal{E}_X: \text{topos}$   
 ② Any  $f: X \rightarrow Y$  induce an ess geom morph

$$\begin{array}{ccc} \mathcal{E}_X & \xrightleftharpoons[\text{f}_*]{\text{f}^*} & \mathcal{E}_Y \\ \text{étale.} \rightsquigarrow (\mathcal{E}_X)_{F_X} & \xrightleftharpoons[\pi_*]{\pi^*} & \mathcal{E}_Y \end{array}$$

- where
- $f^*$ : pullback
  - $f_! = f \circ (-)$
  - $\pi_* = f_*$ : "objects of sections"  
or " $(\pi_* \mathcal{E})_y = \pi_{f(x)=y} \mathcal{E}_x$ "

exer slice Gro top is Gro top.

Hint first prove  $\text{PSh}(\mathcal{E})_{F_X} \cong \text{PSh}(\mathcal{E}_X)$

obs If  $f: \text{epi}$ , then  $f^*$ : faithful (exer).  
 If  $f: \text{mono}$ , then  $f_*$ : fully faithful.

Def  $\mathcal{E} \xrightleftharpoons[\text{f}_*]{\text{f}^*} \mathcal{F}$  is an embedding if  $f_! = f \cdot f_*$  (i.e. lex loc)

More Generally The factorization (cf) epi-mono factorization

ex.4  $G \in \mathcal{E}$  lex comonad.  $\rightsquigarrow \text{coAlg}_G(\mathcal{E}): \text{topos}$

$$\rightsquigarrow p: \mathcal{E} \xrightleftharpoons[\pi_*]{\pi^*} \text{coAlg}_G(\mathcal{E}) \quad \begin{array}{l} \pi^* = \text{forget} : \text{lex by } G\text{-lex} \\ \pi_* = \text{cofree} \end{array} \quad \begin{array}{l} \text{subj.} \\ \text{geom} \\ \text{morph.} \end{array}$$

$G = f \cdot f^*$

Thm  $\mathcal{E} \xrightleftharpoons[\text{f}_*]{\text{f}^*} \mathcal{F}$  surj  $\iff$  comonadic

proof Beck's Thm +  $\mathcal{E}$ .

ex. 5  $\mathcal{E}: \text{topos}, 1 \xrightarrow{t} \Omega$

Def LT topology on  $\mathcal{E}$  is a morph  $j: \Omega \rightarrow \Omega$   
 s.t.  $j^2 = j, jt = t, j(- \wedge -) = j(-) \wedge j(-)$   
 "kernel"

$\rightsquigarrow \text{Sub}(\mathcal{E}) \cong \mathcal{E}(E, \Omega)$

$$\begin{array}{ccc} \cup A \mapsto \bar{A} & \cup j & \\ \text{closure operator.} & & \end{array}$$

Def  $A \rightarrow E$  dense if  $\bar{A} = E$  (if  $j \circ \chi_A = \chi_E$ .)

•  $F \in \mathcal{E}$   $j$ -sheaf if  $A \rightarrow F$   
 $\rightsquigarrow \text{Sh}_j(\mathcal{E}) \xrightarrow{\text{lex}} \mathcal{E}$

Fact

- $\exists a: \text{lex}$  "sheafification"
- $\forall \text{emb}(\mathcal{F} \xrightarrow{\mathcal{E}} \mathcal{E}) \exists j$  on  $\mathcal{E} \quad \mathcal{F} \cong \text{Sh}_j(\mathcal{E})$
- For  $\mathcal{E} = \text{PSh}(\mathcal{C})$   
 LT top  $\iff$  Gro top on  $\mathcal{C}$   
 by " $S \rightarrow \mathcal{C}$  is dense iff covering"  
 + cocompact (later)

Example:  $\text{pts top sp} * \xrightarrow{x} X$   
 $\rightsquigarrow \text{Set} \xleftarrow{x_* = \text{stalk}_x} \perp \xrightarrow{x_* = \text{stalk}_x} \text{Sh}(X)$

Def: A pt of  $\mathcal{E}$  is a geom mor  $\text{Set} \rightrightarrows \mathcal{E}$ .  
 •  $\mathcal{E}$  has enough pts if stalks jointly detects isom.

Want to understand pts of Gro topoi  $\text{Sh}(\mathcal{C}, \tau)$  using this presentation (or geom into).

II A Ubiquitous Construction (cat theory chap 1)

Def  $\mathcal{C} \xrightarrow{F} \mathcal{E}$  left Kan ext of  $F$  along  $K$   
 $K: \mathcal{D} \rightarrow \mathcal{C}$  is a pair  $[K, F: \mathcal{D} \rightarrow \mathcal{F}]$   
 $\mathcal{O}: \mathcal{F} \rightarrow \mathcal{K}, F \cdot K$   
 Universal among such  
 $(\Leftrightarrow) \mathcal{E}^{\text{cod}}(K, F, G) \simeq \mathcal{E}^{\mathcal{E}}(F, G \circ K)$

Exercise If  $K: \text{fully faithful}$ ,  
 $\mathcal{O}: \mathcal{F} \Rightarrow K, F \cdot K$  isom

Pointwise Kan ext if  $\mathcal{E}: \text{cocomp}$ ,  $K, F$  always exists by  $\mathcal{E} \text{ small}$   
 $\text{colim}[K, d] \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{E} =: K, F(d)$

ex " $\otimes$ -Hom" adjunction  
 $\mathcal{C} \xrightarrow{F} \mathcal{E} : \text{cocomplete}$   
 $\mathcal{C}^{\text{op}} \rightarrow \text{Set} \xrightarrow{\text{Psh}(\mathcal{C})} \mathcal{E}$   
 $\text{Hom}_{\mathcal{E}}(F, -)$   
 "left  $\mathcal{E}$ -mod" / "right  $\mathcal{E}$ -mod"

when  $\mathcal{E} = \text{Set}$  (make sense for  $\forall \mathcal{E}$  since  $\mathcal{E}$ : "tensorred"/sets)

$\mathcal{F}, F(G) \simeq \text{colim } Fc \simeq \coprod_c (Gc \times Fc) / ((x, \alpha_x y) \sim (\alpha^* x, y))$   
 $x \in G(c) \Leftrightarrow \exists c' \xrightarrow{\alpha} c$   
 $G \otimes F$   
 $\mathcal{C} \xrightarrow{G} \mathcal{C}' \xrightarrow{F} \mathcal{C}$   
 $\mathcal{C}' \xrightarrow{\alpha} \mathcal{C}$   
 $G \xrightarrow{\alpha^*} G'$

Rem if  $\mathcal{C}^{\text{op}} \xrightarrow{G} \text{Set}$   
 instead  $\downarrow \downarrow$   
 $F \in \text{Set}^{\mathcal{C}}$   
 $F \otimes G$

$\mathcal{F}, \mathcal{F}(e) \xrightarrow{\text{Psh}(\mathcal{C})} \text{Hom}_{\mathcal{E}}(Fc, e) =: \text{Hom}_{\mathcal{E}}(F, -)$

$\text{Hom}_{\mathcal{E}}(F, -)$

$\text{Hom}_{\mathcal{E}}(G \otimes F, e) \simeq \{Gc \times Fc \xrightarrow{\alpha} e\}_{c \in \mathcal{C}}$  : natural  
 $\downarrow \simeq$   
 $\text{Hom}_{\text{Psh}(\mathcal{C})}(G, \text{Hom}_{\mathcal{E}}(F, e)) \simeq \{Gc \rightarrow \text{Hom}_{\mathcal{E}}(Fc, e)\}_{c \in \mathcal{C}}$  : natural

Thm  $\mathcal{E} \text{ small } \mathcal{E} \text{ cocomp} \left[ \text{Fun}(\mathcal{C}, \mathcal{E}) \xrightarrow{\mathcal{F}, F} \text{Fun}(\text{Psh}(\mathcal{C}), \mathcal{E}) \right]$   
 $\downarrow \downarrow$   
 $\text{equiv}$

id by  $\mathcal{F}: \text{fully faithful}$ .  
 $G \rightarrow \text{id}$  by  $\downarrow \downarrow$   
 $\mathcal{F} \downarrow \downarrow \mathcal{F} \downarrow \downarrow$   
 $\mathcal{F} \downarrow \downarrow \mathcal{F} \downarrow \downarrow$

when  $\mathcal{E} = \text{Psh}(\mathcal{C})$ . (L pres. ptwise Kan ext).  
 $\mathcal{F}, \mathcal{F}(e) \xrightarrow{\text{Hom}_{\text{Psh}(\mathcal{C})}} (Fc, -) = e \vee c \rightsquigarrow \mathcal{F}, \mathcal{F} = \text{id}$   
 formula  $P \simeq (\mathcal{F}, \mathcal{F})/P \simeq \text{colim}_{(c \rightarrow P) \in \mathcal{F}} Fc$  "codensity"

III  $\mathcal{E} \simeq \text{Psh}(\mathcal{C})$ .

Def When  $- \otimes F: \text{lex}$  we call  $F$  flat

Thm  $\text{Fun}^{\text{flat}}(\mathcal{C}, \mathcal{E}) \simeq \text{Fun}^{\text{lex}}(\text{Psh}(\mathcal{C}), \mathcal{E}) \simeq \text{Topos}(\mathcal{E}, \text{Psh}(\mathcal{C}))$   
 $\mathcal{F} \xrightarrow{\text{F}}$

we cheated  $\text{flatness}$  in terms of  $F$  itself.  
 want to characterize  $F$  itself.

Def  $(\mathcal{F})^{\text{op}}$ : cat of elts  
 $\downarrow \downarrow \mathcal{F} \xrightarrow{\text{Psh}(\mathcal{C})} \mathcal{C}^{\text{op}}$   
 $\downarrow \downarrow \mathcal{F} \xrightarrow{\text{Set}^{\mathcal{C}}}$   
 $\text{obj}: a \in Fc \leftrightarrow \mathcal{C} \rightarrow F$   
 $\text{mor} (c, a) \xrightarrow{f} (c', a')$   
 $f: c \rightarrow c' \text{ s.t. } a \xrightarrow{Ff} a'$   
 $\mathcal{C} \rightarrow \mathcal{C}'$

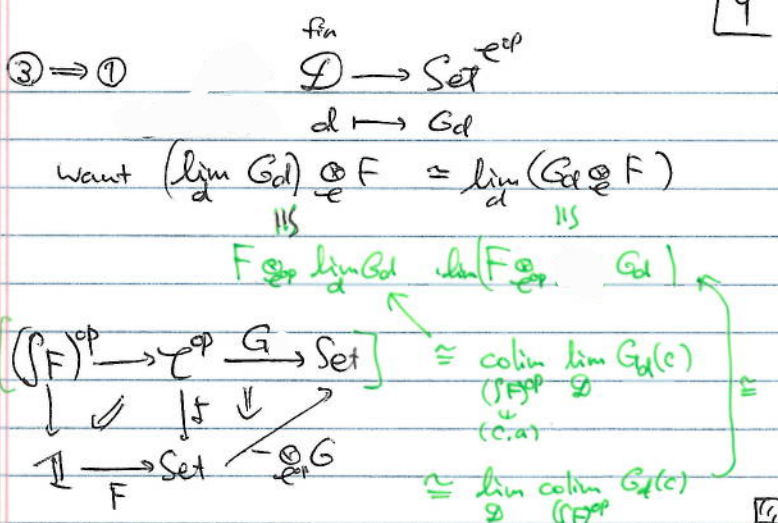
• A small cat  $\mathcal{A}$  is filtered  $\Leftrightarrow \forall \mathcal{D} \rightarrow \mathcal{A}$  finite diagram ( $\mathcal{D}$  can be  $\emptyset$ )  
 $\exists$  cocone on it

Thm For  $\mathcal{C}: \text{finite comp. } \mathcal{C} \xrightarrow{F} \text{Set}$ , TFAE

- ①  $F: \text{flat}$
- ②  $F: \text{lex}$
- ③  $(\mathcal{F})^{\text{op}}: \text{filtered}$ .

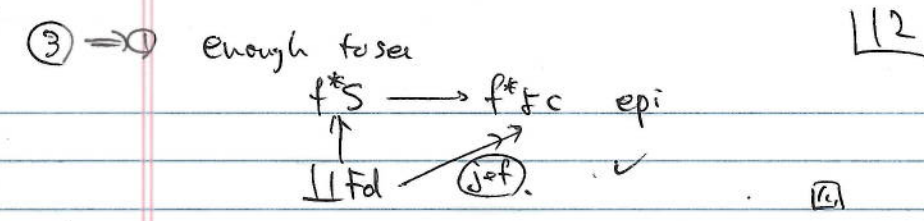
proof ①  $\Rightarrow$  ②  $\mathcal{C} \xrightarrow{F} \text{Set} = \text{lex}$   
 $\downarrow \downarrow \mathcal{F} \xrightarrow{\text{Psh}(\mathcal{C})} \mathcal{C}^{\text{op}}$   
 $\downarrow \downarrow \mathcal{F} \xrightarrow{\text{Set}^{\mathcal{C}}}$

②  $\Rightarrow$  ③  $\mathcal{D} \rightarrow (\mathcal{F})^{\text{op}} \xrightarrow{\text{Psh}(\mathcal{C})} \mathcal{C}^{\text{op}}$   
 $(c_i, a_i \in Fc_i) \xrightarrow{\mathcal{F}} c \xrightarrow{\mathcal{F}} \text{colim } c_i = c$   
 $\downarrow \text{cocone}$   
 $(c, a)$   
 $a_i \rightarrow a$

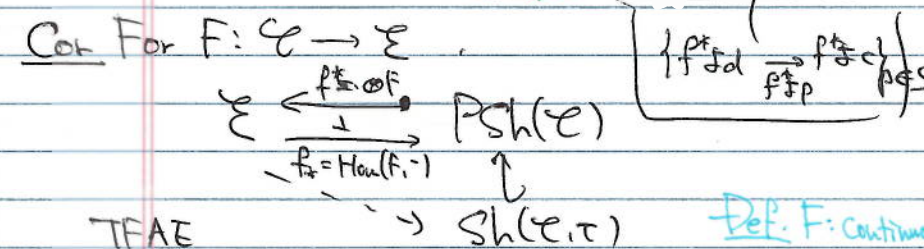
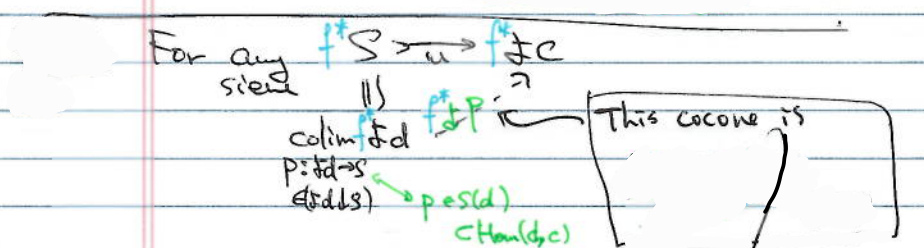
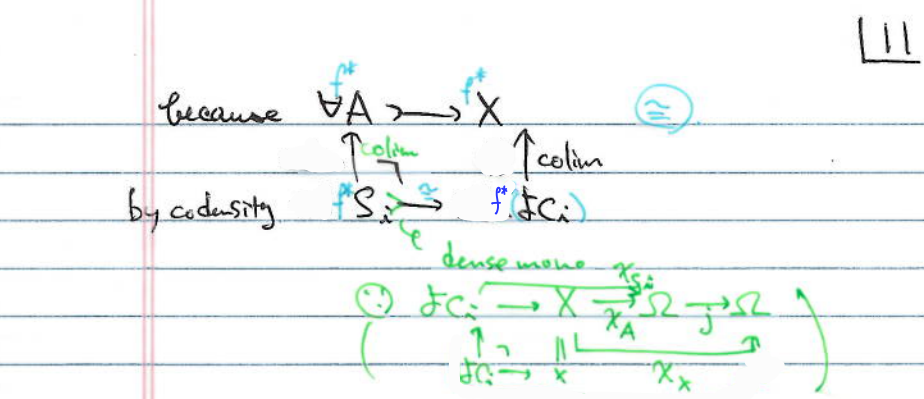


Remark 11.1 finite completeness is dropped, ① ⇔ ③ ⇒ ②

② Set can be replaced by  $\forall \text{cocomp topos } \mathcal{E}$  ("filtered" interpreted internally to  $\mathcal{E}$ )



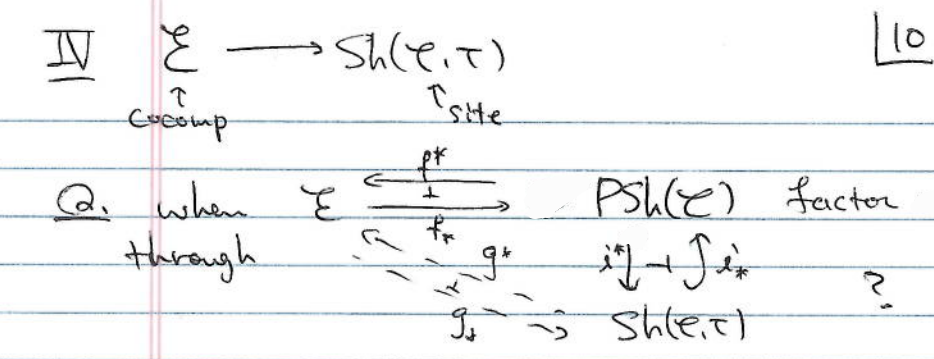
upshot  $\text{Topos}(\mathcal{E}, \text{Sh}(\mathcal{C}, \tau)) \cong \text{Fun}^{\text{cont, flat}}((\mathcal{C}, \tau), \mathcal{E})$



- TFAE
- $f$  factors through  $\text{Sh}(\mathcal{C}, \tau)$
  - Cocone  $\{ f^* \text{ jef} \}_{p \in S}$  is a colim diagram in  $\mathcal{E}$
  - $\{ \text{ jef} \}_{p \in S}$  is jef.

put ① ⇔ ② colim  $f^* \text{ jef} \cong f^* \text{ colim jef}$

② ⇒ ③  $f^* \text{ jef} \xrightarrow{f^* u} f^* \text{ jef}$  by Lem ③



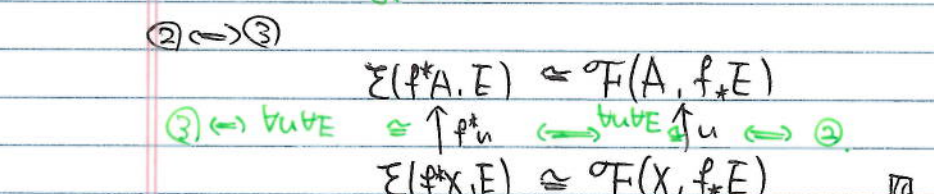
More generally Lem  $\mathcal{F}_i$ : topos w/ LT topology  $\Omega \xrightarrow{j} \Omega$

⇒ TFAE for  $f: \mathcal{E} \xrightarrow{f^*} \mathcal{F}_i$

- $f^*$  factor through  $i$
- $\forall E \in \mathcal{E} f_* E$  is  $j$ -sheaf
- $\forall$  dense mono  $A \xrightarrow{u} X$  in  $\mathcal{F}_i$ ,  $f^* u$  is iso.

proof ① ⇒ ② ✓

② ⇒ ①  $\lambda_* i^* f_* E \cong f_* E$ ,  $g^* := f^* i_*$



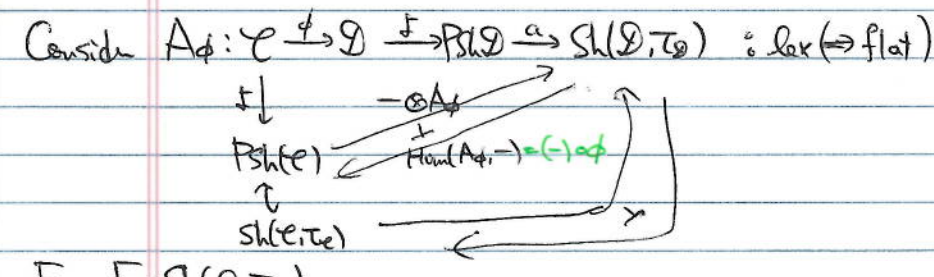
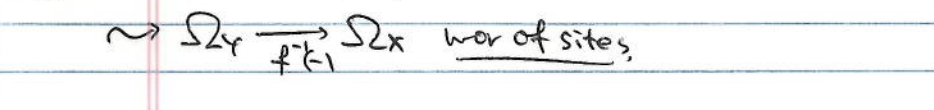
Observation When  $\mathcal{F}_i = \text{PSh}(\mathcal{C})$  (Gro topology) enough to check for covering sieves  $S \rightarrow \text{ jef}$

V. Mor of sites

Assume  $(\mathcal{C}, \tau_{\mathcal{C}}), (\mathcal{D}, \tau_{\mathcal{D}})$  sites, finite complete.

Def  $\phi: \mathcal{C} \rightarrow \mathcal{D}$  functor is a mor of sites if

- lex
- pres covers. i.e.  $\forall$  cov sieve  $S = \{c' \rightarrow c\}$  of  $\mathcal{C}$ ,  $\phi S = \{\phi c' \rightarrow \phi c\}$  generates cov. sieve of  $\mathcal{D}$ .



For  $F \in \text{Sh}(\mathcal{D}, \tau_{\mathcal{D}})$

$$\text{Hom}(A_\phi, -) = c \mapsto \text{Hom}_{\text{Sh}}(a \circ f \circ \phi(c), F) \cong \text{Hom}_{\text{PSh}}(f \circ \phi(c), F) \cong F \circ \phi(c)$$

$A_\phi$  conti by fact  $\{d_i \rightarrow d\}$  generates cov sieve of  $\mathcal{D}$  iff  $\{a d_i \rightarrow a d\}$  is jef (SGLI.7)

Then  $\phi: \mathcal{C} \rightarrow \mathcal{D}$  induce geom mor  $f_\phi: \text{Sh}(\mathcal{D}, \tau_{\mathcal{D}}) \xrightarrow{(-) \circ \phi} \text{Sh}(\mathcal{C}, \tau_{\mathcal{C}})$