1

Operads • $(\mathcal{M}, \otimes, I)$ sym mon. closed cat. cocomplete.

~~Def~~ • Σ : category ob : $\{1, 2, \cdots, n\}$ $(n \geq 1)$ "symmetric groupoid".

mor : bijection. $\Sigma_n := \mathrm{Aut}_\Sigma(n)$.

• $\mathbb{N}$: discrete category with obj $n (\geq 1)$.

(reduced) ~~symmetric sequence~~

~~Σ-module~~ (sym. seq.) $=$ functor $\Sigma^{op} \to \mathcal{M}$. $M :$

"$M(0) = 0$"

$= \{ M(\boxed{n}) \in \mathcal{M} \} \curvearrowleft \Sigma_n$. arity

---

Def ① of operad

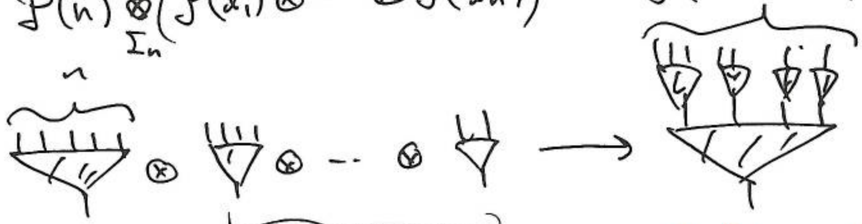
Operad in $\mathcal{M}$ is $(\mathcal{P}, \gamma, \eta)$.

$\mathcal{P} \in \mathcal{M}^{\Sigma^{op}}$, γ : composition map, η : identity.

• $\mathcal{P}(n)$: space of n-ary operations $\ni$  $\curvearrowright \Sigma_n$.

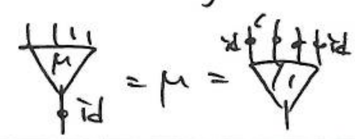
• $\gamma : \mathcal{P}(n) \otimes_{\Sigma_n} (\mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_n)) \longrightarrow \mathcal{P}(i_1 + \cdots + i_k)$.

"$\Sigma_{i_1} \times \cdots \times \Sigma_{i_k} \subset \Sigma_{i_1 + \cdots + i_k}$"
"equivariant"
"associative".



$(\mu ; \nu_1, \cdots, \nu_n) \longmapsto \mu \circ (\nu_1, \cdots, \nu_n)$.

• $\eta : I \longrightarrow \mathcal{P}(1)$ (or $\mathrm{id} \in \mathcal{P}(1)$). identity



mor of operad $f : \mathcal{P} \to \mathcal{Q}$: nat. tr. comm. w/ γ & η.

◎ Want to pack the information into more concise form.

e.g. $\mathrm{Ass}(n) = I \cdot \Sigma_n$. copower

$\mathrm{Com}(n) = I$.

$\mathrm{End}(V)$.

Def ② $\mathcal{M}^{\Sigma^{op}} \longrightarrow \mathrm{End}(\mathcal{M})$

$\{M(n)\} \longmapsto \tilde{M}$: schur functor of M.

②

$$\widetilde{M}(V) := \int^{n\in\Sigma} \underset{\mathcal{M}^{\Sigma^{op}}}{\underset{\cap}{\underline{M(n)}}} \otimes \underset{\mathcal{M}^{\Sigma}}{\underset{\cap}{\underline{V^{\otimes n}}}} \quad \left(= \coprod_{n\geq 1} M(n) \underset{\Sigma_n}{\otimes} V^{\otimes n} \right).$$

space of operations & its inputs

ⓑ $$M\otimes N(n) := \int^{i,j\in\Sigma} \Sigma(i+j, n)\otimes M(i)\otimes N(j)$$ Day convolution.

$$\left(= \coprod_{i+j=n} (M(i)\otimes N(j)) \underset{\Sigma_i\times\Sigma_j}{\otimes} \Sigma_n \right).$$

pair of operations in M & N.

ⓒ $$\boxed{M\circ N} := \int^{k\in\Sigma} M(k)\otimes N^{\otimes k}$$ composite product.

$$\left(\rightsquigarrow M\circ N(n) = \coprod_{k\in\Sigma} M(k) \underset{\Sigma_k}{\otimes} N^{\otimes k} = \int^{k\in\Sigma}\int^{i_1,\dots,i_k\in\Sigma} \Sigma(i_1+\dots+i_k, n)\otimes M(k) \otimes N(i_1)\otimes\cdots\otimes N(i_k). \right.$$

Composable tuple.

$\in M(3)\otimes N^{\otimes 3}$

ⓓ $$\mathcal{M} \hookrightarrow \mathcal{M}^{\Sigma^{op}}$$
$$M \longmapsto M(n) = \begin{cases} M & (n=1) \\ 0 & (\text{otherwise}). \end{cases}$$

exer. $$\widetilde{M\otimes N} \cong \widetilde{M}\otimes\widetilde{N} : V \longmapsto \widetilde{M}(V)\otimes\widetilde{N}(V).$$
$$\widetilde{M\circ N} \cong \widetilde{M}\circ\widetilde{N} : V \longmapsto \widetilde{M}(\widetilde{N}(V)).$$
$$\widetilde{I} = \mathrm{id}_{\mathcal{M}}$$

~~Schur for constr. induce strong mon. fct.~~

→ Sequence of strong monoidal functor.

$$(\mathcal{M}, \otimes, I) \underset{ⓓ}{\hookrightarrow} (\mathcal{M}^{\Sigma^{op}}, \circ, I) \underset{ⓐ}{\longrightarrow} (\mathrm{End}(\mathcal{M}), \circ, \mathrm{id}_{\mathcal{M}}).$$

$$M \longmapsto (M\otimes -).$$

~~$\rightsquigarrow \mathrm{Mon}(\mathcal{M}) \longrightarrow \mathrm{Mon}(\mathcal{M}^{\Sigma^{op}})$~~

$$\rightsquigarrow \mathrm{Mon}(\mathcal{M}, \otimes, I) \longrightarrow \mathrm{Mon}(\mathcal{M}^{\Sigma^{op}}, \circ, I) \longrightarrow \mathrm{Mon}(\mathrm{End}(\mathcal{M}))$$

Def② $\| \; \mathrm{Op}(\mathcal{M})$ $\qquad$ $\| \; \mathrm{Monad}(\mathcal{M})$.

exer. check Def ① $\Leftrightarrow$ Def ②.

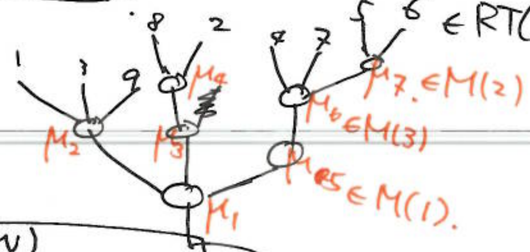
Def ②. $\mathcal{P} \in \mathcal{M}^{\Sigma^{op}}$

$\begin{cases} \gamma : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P} \\ \eta : I \longrightarrow \mathcal{P} \end{cases}$ assoc & unital.

~~algebs~~ ∇: $\mathcal{P}$-alg = alg / monad $\Leftrightarrow \mathcal{P} \to \mathrm{End}_V$.

Q ~~...~~ $\boxed{\text{tree of operations}} \Rightarrow \boxed{\text{single operation}}$
composition

Def. $RT(n) := \{$reduced rooted tree with leaves labelled by $\{1,2,\dots,n\}\}$.
$\forall$ Vert has ≥ 1 incoming edges.



$\in RT(9)$. $\in \mathbb{T}M(9)$.

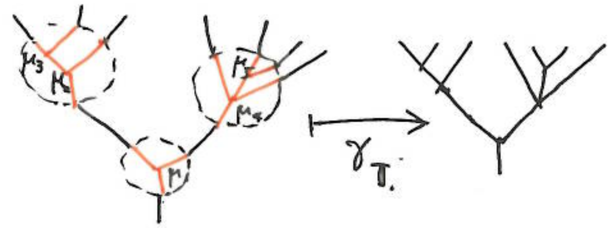
$$\mathbb{T}M(n) = \bigoplus_{t \in RT(n)} \bigotimes_{v \in Vert(t)} M(\# in(v)).$$

$\boxed{in(v) = \{\text{incoming edges}\}}$.

tree of gen. op.

$\mathbb{T} : \mathcal{M}^{\Sigma^{op}} \longrightarrow \mathcal{M}^{\Sigma^{op}}$

$(\mathbb{T}, \gamma_{\mathbb{T}}, \eta_{\mathbb{T}})$ monad by

- 

  $\mathbb{T} \circ \mathbb{T}(M) \longrightarrow \mathbb{T}(M)$.
  tree of trees

- $M \xrightarrow{\eta_{\mathbb{T}}(M)} \mathbb{T}M$.
  $\mu \longmapsto$  corolla.

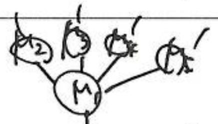
Def ③.

$Op(\mathcal{M}) = \mathbb{T}\text{-alg}(\mathcal{M}^{\Sigma^{op}})$.

$(\mathcal{P}, \mathbb{T}\mathcal{P} \xrightarrow{\gamma} \mathcal{P})$

$\boxed{\text{Def ③} \Leftrightarrow \text{Def ①}}$

- $\mathbb{T}\mathcal{P}(1) \supset I \xrightarrow{\eta} \mathcal{P}(1)$.
  t : trivial tree

- 

  2-leveled tree recover γ in Def ①

Cor. $\mathcal{M}^{\Sigma^{op}} \underset{\text{forget}}{\overset{\mathcal{T}}{\rightleftarrows}} Op(\mathcal{M})$ ($\perp$)

$\mathcal{T}M$: free operad on $\mathcal{M}$.
$= (\mathbb{T}M, \gamma_{\mathbb{T}}(M), \eta_{\mathbb{T}}(M))$.

Def. weight of $\mu \in \mathcal{T}(M)$ ... # of generating operations in μ. $\mathcal{T}(M)^{(k)} = \{\mu \in \mathcal{T}(M) \mid \text{weight} \leq k\}$

Operad $\mathcal{P}$ is coaugmented $\Leftrightarrow$ $\exists \mathcal{P} \longrightarrow I$ mor of operads $\left(\eta \uparrow \curvearrowleft \varepsilon, \; I \right)$.

$\boxed{\text{nonsymmetric operad --- replace } \overset{\text{RT} \quad \text{PRT}}{\boxed{I \text{ by } \mathbb{N}}}}$ e.g. As.

Cooperads [from now on $\mathcal{M} = \mathrm{Vect}_k$ or grVect_k or dgVect_k (k: field of char $= 0$)]

- $\forall n$ $k[\Sigma_n]$ semisimple. $\Rightarrow$ $\forall k[\Sigma_n]$-mod is projective.
- $(-)_{\Sigma_n} \longrightarrow (-)^{\Sigma_n}$ is isom.
- $\oplus = \times = \sqcup$
- linear dual.

Def. ① $\mathcal{C}$: cooperad in $\mathcal{M}$ is a comonoid in $(\mathcal{M}^{\Sigma^{op}}, \circ, I)$ "how to decompose operations".

$\varepsilon: \mathcal{C} \to I \qquad \Delta: \mathcal{C} \to \mathcal{C} \circ \mathcal{C}$.

② augmented when $\exists \eta: I \to \mathcal{C}$ coop mor.

$$\rightsquigarrow I \oplus \overline{\mathcal{C}}$$
($\uplus$ id)

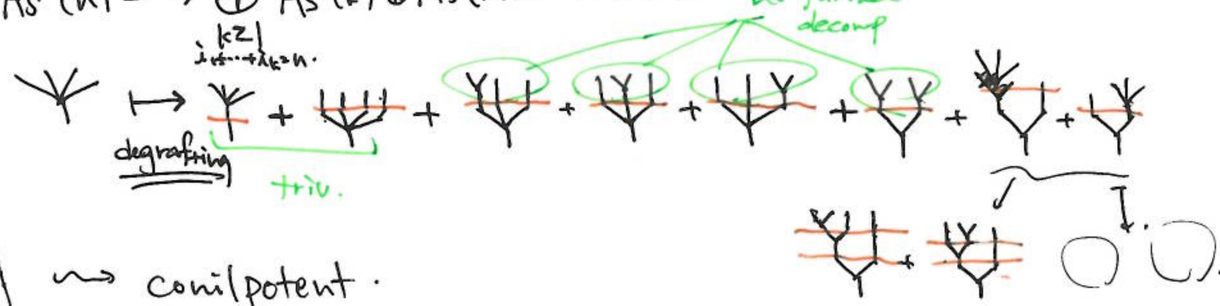
③ $\mathcal{C} = \overline{\mathcal{C}} \oplus I$ is conilpotent if "any sequence of (upward) non-trivial decomp terminates".

e.g. nonsym, $\mathcal{M} = \mathrm{Vect}_k$.

$$As^*_{(n)} = \mathrm{Hom}(As(n), k) = k.$$
($As(n) = k$)

denote $k \xrightarrow{1} k$ generator by $\overset{n}{\Psi}$

$$As^*(n) \longrightarrow \bigoplus_{\substack{k \geq 1 \\ i_1 + \cdots + i_k = n}} As^*(k) \otimes As^*(i_1) \otimes \cdots \otimes As^*(i_k).$$



$\rightsquigarrow$ conilpotent.

④ $M \in \mathcal{M}^{\Sigma^{op}}$ $\rightsquigarrow$ $\mathcal{T}^c M := (\mathbb{T}M, \Delta, \varepsilon, \eta)$.

$\mathbb{T}M \oplus I$ - trivial tree. $= \mathcal{T}^c M$, $I \to \mathcal{T}^c M \to I$

(tree of gen. operations.) degrafting as above. ~~proj~~ onto corolla.

is cofree in conil. coop.

$$\begin{tikzcd} & \mathcal{T}^c M \arrow[d] \\ \underset{\text{conil}}{\mathcal{C}} \arrow[ur, dashed] \arrow[r] & M \end{tikzcd}$$

$\underline{\text{dg}^{(co)}\text{-Operads}} = {}^{(co)}$operads in $(\text{dg-Vect}_k, \otimes, k)$. $\quad$ $\underline{k\text{: field, char} = 0}$.

Koszul sign rule.

- $d_{A\otimes B}(a\otimes b) = d_A a\otimes b + (-1)^{|a|} a\otimes d_B b$.
- $\tau : A\otimes B \longrightarrow B\otimes A$, $\quad a\otimes b \longmapsto (-1)^{|a||b|} b\otimes a$.
- $(\mathrm{Hom}_\bullet(A, B))_n = \prod \mathrm{Hom}(A_\bullet, B_{\bullet+n}) \ni f$
  $\downarrow \partial \qquad\qquad\qquad \downarrow$
  $(\mathrm{Hom}(A, B))_{n-1} \qquad\qquad \ni \partial f$
  $\partial f := [d, f] = d_B f - (-1)^{|f|} f d_A$.

Assume augmented.

~~$\mathcal{P}$ ∈ dgOp. $\mathcal{C}$ ∈ dgcoOp.~~

~~convolution opera~~

~~$\rightsquigarrow \mathrm{Hom}_\Sigma(\mathcal{C}, \mathcal{P}) = \prod_{n\geq 1} \mathrm{Hom}_{\Sigma_n}(\mathcal{C}(n), \mathcal{P}(n))$ ∈ gr Vect~~

$\underline{\text{exer}}$ check.

$(M_\bullet, d_M), (N_\bullet, d_N) \in (\text{dgVect})^{\Sigma\text{-op}}$ — $\boxed{\text{arity \& degree grading.}}$

$\rightsquigarrow (M_\bullet \circ N_\bullet, d_{M\circ N})$ comp prod.

$d_{M\circ N}$ is given by.

$$(\mu; \nu_1, \cdots, \nu_n) \longmapsto (d\mu; \nu_1, \cdots, \nu_n) + \sum_{i=1}^{n} (-1)^{\varepsilon_i} (\mu; \cdots, d\nu_i, \cdots \nu_n)$$

$$(\varepsilon_i = |\mu| + |\nu_1| + \cdots + |\nu_{i-1}|)$$

K sign rule + Leibniz rule

Weight 2 part of

$(\mathbb{T}M)^{(2)} \ni$ μ_2 μ_1 : denote by

$M \underset{(1)}{\circ} M. \longrightarrow \begin{cases} \Delta_{(1)} : \mathcal{C} \to \mathcal{C} \underset{(1)}{\circ} \mathcal{C}. \\ \gamma_{(1)} : \mathcal{P} \underset{(1)}{\circ} \mathcal{P} \to \mathcal{P}. \end{cases}$

$\underline{\text{Twisting morphism}}$.

$\mathcal{P}$: dg op, $\mathcal{C}$: dg coop.

$\rightsquigarrow \mathrm{Hom}_\Sigma(\mathcal{C}, \mathcal{P}) := \prod_{n\geq 1} \mathrm{Hom}_{\Sigma_n}(\mathcal{C}(n), \mathcal{P}(n)). \in \text{dgVect}_k.$

$\qquad\qquad\qquad\qquad\qquad\qquad \cap$
$\qquad\qquad\qquad\qquad\qquad\qquad$ dgVect.

$f, g \in \mathrm{Hom}_\Sigma(\mathcal{C}, \mathcal{P})$

$\rightsquigarrow f \star g$ is defined by.

$$\mathcal{C} \xrightarrow[\Delta]{} (\mathcal{T}^c\mathcal{C}) \twoheadrightarrow \underbrace{(\mathcal{T}^c\mathcal{C})^{(2)}}_{\mathcal{C}\underset{(1)}{\circ}\mathcal{C}} \longrightarrow (\mathcal{T}\mathcal{P})^{(2)} \xrightarrow{\gamma} \mathcal{P}$$

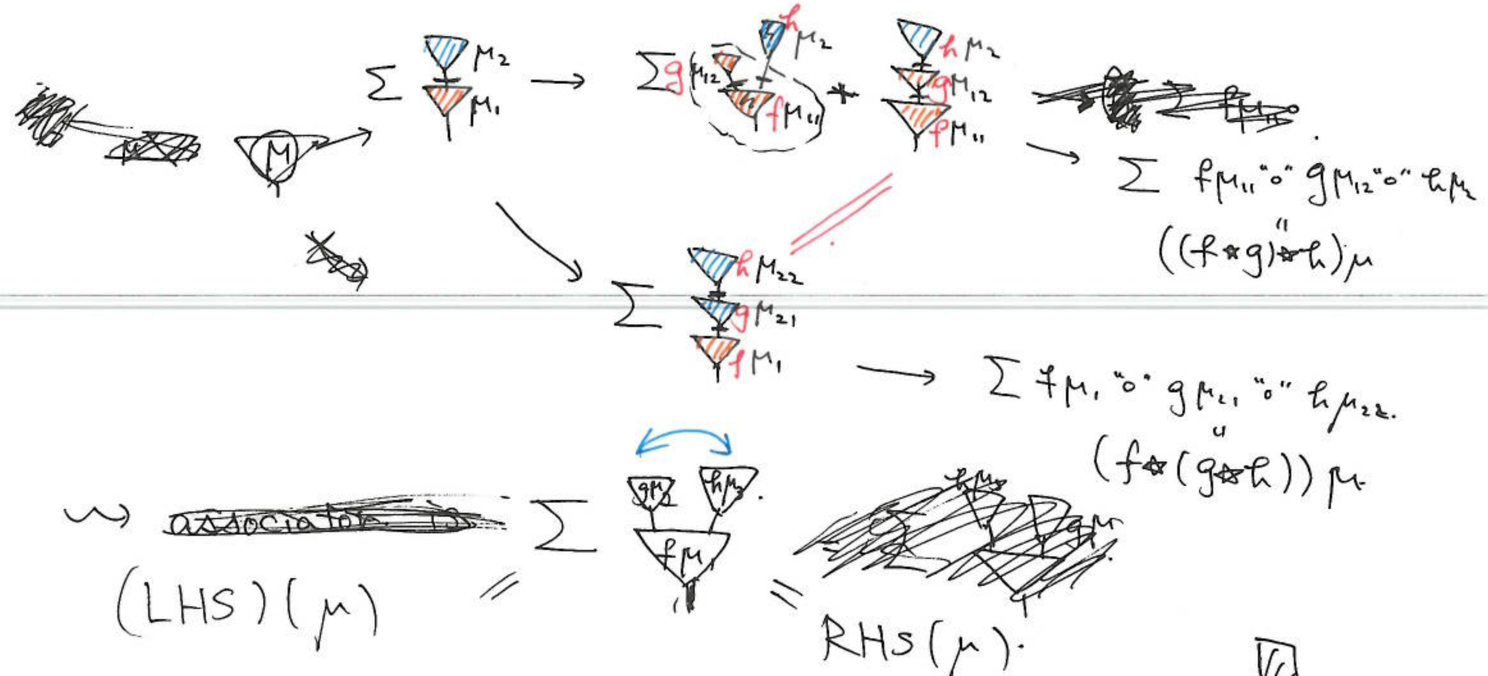
$\mu \longmapsto \sum$ $\mu_2 \in \mathcal{C}$, $\mu_1 \in \mathcal{C}$ $\longmapsto \sum$ $g\mu_2 \in \mathcal{P}$, $f\mu_1 \in$ $\xrightarrow{\gamma} \sum (f\mu_1) \underset{i}{\circ} (g\mu_2)$.

weight 2 part of $\Delta\mu$.

(check : equivariance).

Lem. ★ is a pre-Lie product $\left[\begin{array}{l}(f\star g)\star h - f\star(g\star h)\\ = (f\star h)\star g - f\star(h\star g).\end{array}\right]$

(associator is right commutative).



~~In general,~~

exer. ★ : pre-Lie product. $\rightsquigarrow$ antisymmetrization $[f,g]$

(cf. assoc. alg $\to$ Lie alg).

Def. dg Lie alg $(L_\bullet, [\,], \partial)$. ∂: deg -1 derivation $\partial[x,y] = [\partial x, y] + (-1)^{|x|}[x,\partial y]$, $\partial^2 = 0$

$$\left[\begin{array}{l}\bullet\ [x,y] = -(-1)^{|x||y|}[x,y]\\ \bullet\ [z,[x,y]] = [[z,x],y] + (-1)^{|x||z|}[x,[z,y]]\end{array}\right.$$

i.e. $[z,-]$ derivation.

$$\rightsquigarrow MC(L) = \{\alpha\in L_{-1} \mid \partial\alpha + \tfrac{1}{2}[\alpha,\alpha] = 0\}$$

$$Tw(\mathcal{C},\mathcal{P}) := MC(\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P}), [\,,\,], \partial) = \{\alpha\in\mathrm{Hom}_\Sigma(\mathcal{C},\mathcal{P})_{-1} \mid \partial\alpha + \alpha\star\alpha = 0\}$$

Bar / cobar construction

$$\mathrm{gr.Op}(\mathcal{T}(s^{-1}\overline{\mathcal{C}}), \mathcal{P}) \cong (\mathrm{Hom}_{\Sigma}(\overline{\mathcal{C}}, \overline{\mathcal{P}}))_{-1} \cong \mathrm{gr.coOp}(\mathcal{C}, \mathcal{T}^c(s\overline{\mathcal{P}}))$$

$$\cup \qquad\qquad \cup \ \mathrm{MC} \qquad\qquad \cup$$

$$\mathrm{dgOp}(\Omega\mathcal{C}, \mathcal{P}) \cong \mathrm{Tw}(\mathcal{C}, \mathcal{P}) \cong \mathrm{dgcoOp}(\mathcal{C}, B\mathcal{P})$$

representable !!

$\boxed{B : \{\text{coaug. dg Op}\} \to \{\text{conil dg coop}\}}$

Q [redacted] : underlying graded cooperad is $\mathcal{T}^c(s\overline{\mathcal{P}})$.

$B\mathcal{P}$

$\left[\begin{array}{l} d_1 : \text{diff. induced} \\ \text{by diff of } \mathcal{P} \end{array}\right]$

$\mathbb{K}s^{-1}$ sym seq.
$\mathbb{K}s$: conc. in arity 1. deg 1.
$M \in \mathrm{grVect}$. (dg).
$\to sM = \mathbb{K}s \otimes M$ susp
$s^{-1}M = \mathbb{K}s^{-1} \otimes M$ desusp.

"twist" the differential by another differential d_2 on graded operad $\mathcal{T}^c(s\overline{\mathcal{P}})$:

~~cf. want to "extend" bar construction in group coh. : G group.~~

~~$[g_1|g_2|\cdots|g_n] \mapsto \sum (-1)[g_1|\cdots|g_i g_{i+1}|\cdots|g_n]$~~

~~recall bar resol in Gcoh.~~

~~G fin gp. ⇒ projective (free) resol of $\mathbb{Z}[G]$-mod.~~

original bar resol.

$[g_0|\cdots|g_n]$ deg n.

$\downarrow$

$\sum \pm [g_0|\cdots|g_i g_{i+1}|\cdots|g_n]$

deg n-1.

role of suspension

$$d_2 : \mathcal{T}^c(s\overline{\mathcal{P}}) \twoheadrightarrow \mathcal{T}^c(s\overline{\mathcal{P}})^{(2)} \cong (\mathbb{K}s \otimes \overline{\mathcal{P}}) \circ_{(1)} (\mathbb{K}s \otimes \overline{\mathcal{P}})$$

$$\downarrow \mathrm{id} \otimes \tau \otimes \mathrm{id}$$

$$(\mathbb{K}s \otimes \mathbb{K}s) \otimes (\overline{\mathcal{P}} \circ_{(1)} \overline{\mathcal{P}})$$

$$\downarrow s \otimes s \mapsto s \ \otimes \gamma_{(1)}$$

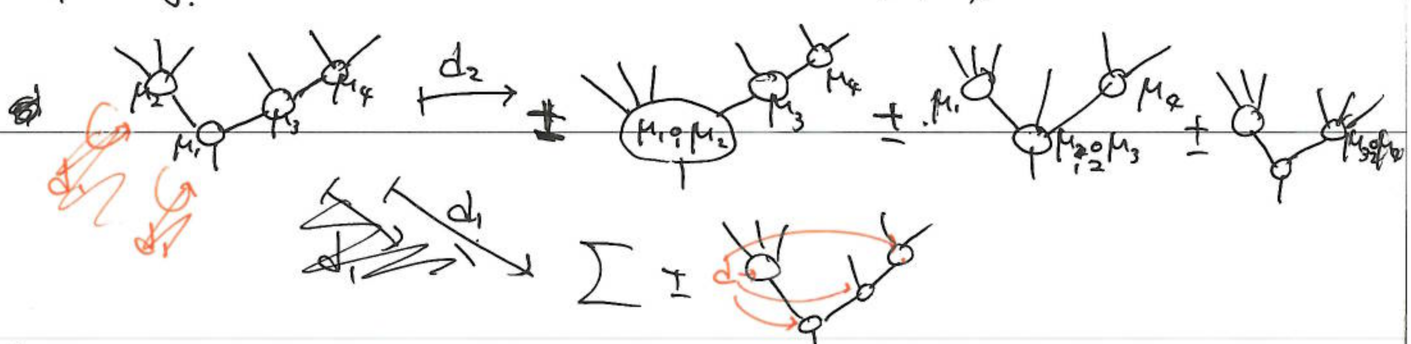
$$\mathbb{K}s \otimes \overline{\mathcal{P}} = s\overline{\mathcal{P}}$$

$$\uparrow$$

$$\mathcal{T}^c(s\overline{\mathcal{P}})$$

lifts to unique derivation (comm. w/ Δ).

explicitly.



$d = d_1 + d_2$. **fact** $d^2 = 0$.

$\Omega : \{\text{aug dg coop}\} \to \{\text{aug dg op}\}$:

$\mathcal{C} \mapsto \Omega\mathcal{C}$ cobar construction.

"twist" the free dg operad $(\mathcal{T}(s^{-1}\bar{\mathcal{C}}), d_1)$ by another diff.

$$\begin{tikzcd}
d_2 : \mathbb{K}s^{-1}\otimes\bar{\mathcal{C}} \arrow[r, "(s^{-1}\mapsto s^{-1}\otimes s^{-1})\otimes\Delta_{(1)}"'] \arrow[dd, dashed] & \mathbb{K}s^{-1}\otimes\mathbb{K}s^{-1}\otimes\bar{\mathcal{C}}\circ_{(1)}\bar{\mathcal{C}} \arrow[d, "\tau", "\cong"'] \\
 & (\mathbb{K}s^{-1}\otimes\bar{\mathcal{C}})\circ_{(1)}(\mathbb{K}s^{-1}\otimes\bar{\mathcal{C}}) \cong \mathcal{T}(s^{-1}\bar{\mathcal{C}})^{(2)} \arrow[d, phantom, "\cap"] \\
\mathcal{T}(s^{-1}\bar{\mathcal{C}}) \arrow[r, dashed, "\exists!\ \text{extend to derivation}"] & \mathcal{T}(s^{-1}\bar{\mathcal{C}}).
\end{tikzcd}$$

(i.e. commute w/ composition).

$d = d_1 + d_2$

<u>fact</u> $d^2 = 0$.

<u>Thm.</u> $\mathrm{dgOp}(\Omega\mathcal{C}, \mathcal{P}) \underset{①}{\cong} \mathrm{Tw}(\mathcal{C}, \mathcal{P}) \underset{②}{\cong} \mathrm{dgcoOp}(\mathcal{C}, B\mathcal{P})$.

∵ ① $\mathrm{dgOp}(\Omega\mathcal{C},\mathcal{P}) \subset \mathrm{grOp}(\Omega\mathcal{C}, \mathcal{P})$

$\Omega\mathcal{C} = \mathcal{T}(s^{-1}\bar{\mathcal{C}})$ — free.

$\mathrm{grOp}(\Omega\mathcal{C}, \mathcal{P}) \cong \left\{ \text{degree } (-1) \text{ morphism } \bar{\mathcal{C}} \xrightarrow{\alpha} \mathcal{P} \right\}$.

this commutes w/ differential: on generators

$$\begin{tikzcd}
\mu \in s^{-1}\bar{\mathcal{C}} \arrow[r, "\alpha"] \arrow[d, "d_1+d_2"'] & \mathcal{P} \arrow[d, "d_{\mathcal{P}}"] \\
\mathcal{T}(s^{-1}\mathcal{C}) \arrow[r, "\alpha"] & \mathcal{P}
\end{tikzcd}$$

① ①

αd_1

αd_2 ⇝ $\alpha(\mu_2)$, $\alpha(\mu_1)$ ⇝ $\alpha \star \alpha + \boxed{d_{\mathcal{P}}\alpha \pm \alpha d_{\mathcal{C}} = \partial\alpha} = 0$

$\iff \alpha \in \mathrm{Tw}(\mathcal{C}, \mathcal{P})$.

② similar.

□

~~Homotopy transfer theorem.~~ $\Omega\mathscr{C}$-algebra str. is ho. inv.

(V, d_V) : homotopy retract of (W, d_W).

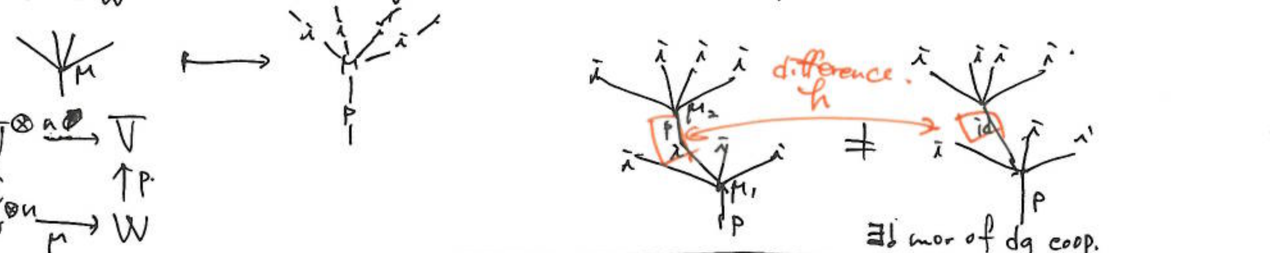
i.e. $h \circlearrowright (W, d_W) \underset{i}{\overset{p}{\rightleftarrows}} (V, d_V)$ $\left[\begin{array}{l} i, p \text{ chain map.} \\ i : \text{q-iso.} \\ id_W - ip = d_W h + h d_W. \end{array}\right.$

(weaker than ho. equ.
/field $\forall$ qis i can be extended to ho retract data)

◎ This data induces a mor of dg cooperads

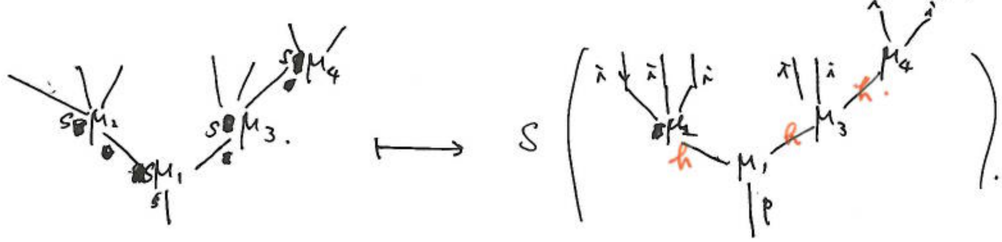
$$B\,End_W \longrightarrow B\,End_V$$

$End_W \longrightarrow End_V$ is not a map of operads.



$$\begin{array}{ccc} V^{\otimes n} & \xrightarrow{\ \ } & V \\ {\scriptstyle i^{\otimes n}}\downarrow & & \uparrow{\scriptstyle p} \\ W^{\otimes n} & \xrightarrow{\mu} & W \end{array}$$

$B\,End_W = \mathscr{T}^c(s\,End_W) \xrightarrow{\text{mor of sym seq.}} s\,End_V \rightleftarrows \mathscr{T}^c(s\,End_V) = B\,End_V.$



exer. this commute with differential.

$\rightsquigarrow \{\Omega\mathscr{C}\text{-alg str. on } W\} = dgOp(\Omega\mathscr{C}, End_W) \cong dgcoOp(\mathscr{C}, B\,End_W).$

if ho equ.

$\{\Omega\mathscr{C}\text{-alg str. on } V\} = dgcoOp(\Omega\mathscr{C}, End_V) \cong dgcoOp(\mathscr{C}, B\,End_V)$

~~When $\mathscr{C} = \mathscr{P}$~~

$\rightsquigarrow$ We want a "resolution" Ω ⓪ $\xrightarrow{\sim} \mathscr{P}$.

① General one : bar-cobar resol.

(but huge).

$$\Omega B\mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

②. small one : Koszul resol.

$$\Omega \mathcal{P}^{i} \xrightarrow{\sim} \mathcal{P}$$

Twisted composite product

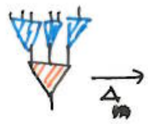
$\mathcal{C}$: aug. dg coop, $\mathcal{P}$: aug. dg op, $\alpha \in (\mathrm{Hom}_\Sigma(\mathcal{C}, \mathcal{P}))_{-1}$

$\mathcal{C} \cong \overline{\mathcal{C}} \oplus I$, $\mathcal{P} \cong \overline{\mathcal{P}} \oplus I$.

$\hookrightarrow$ Comp prod $(\mathcal{P}\circ\mathcal{C}, d_{\mathcal{P}\circ\mathcal{C}})$. | $(\mathcal{C}\circ\mathcal{P}, d_{\mathcal{C}\circ\mathcal{P}})$.

"twist" the differential by.

$d^{l}_{\alpha} : \mathcal{P}\circ\mathcal{C} \to \mathcal{P}\circ\mathcal{C}\circ$



$\rightsquigarrow$ total differential $d_{\mathcal{P}\circ\mathcal{C}} + d^{l}_{\alpha} =: d_\alpha$. $\quad d_\alpha := d_{\mathcal{C}\circ\mathcal{P}} + d^{r}_{\alpha}$

<u>Lem</u> $d_\alpha^2 = d^{l\,(\text{or } r)}_{\partial\alpha + \alpha\star\alpha} = 0$ iff $\alpha \in \mathrm{Tw}(\mathcal{C}, \mathcal{P})$.

<u>Def.</u> $\alpha \in \mathrm{Tw}(\mathcal{C}, \mathcal{P})$ twisted composite prod

$\mathcal{C} \circ_\alpha \mathcal{P} := (\mathcal{C}\circ\mathcal{P}, d_\alpha)$

$\mathcal{P} \circ_\alpha \mathcal{C} := (\mathcal{P}\circ\mathcal{C}, d_\alpha)$.

$\pi : B\mathcal{P} \to \mathcal{P}$ and $\iota : \mathcal{C} \to \Omega\mathcal{C}$. univ. tw. mor.

<u>Lem②</u> $\left.\begin{array}{l}\mathcal{P}\circ_\pi B\mathcal{P},\ B\mathcal{P}\circ_\pi\mathcal{P} \\ \mathcal{C}\circ_\iota\Omega\mathcal{C},\ \Omega\mathcal{C}\circ_\iota\mathcal{C}\end{array}\right]$ acyclic.

Today's blackbox

Lem① (comparison lemma).

$$\begin{array}{ccc} \mathcal{P} & \xleftarrow{\alpha} & \mathcal{C} \\ f\downarrow & \circlearrowleft & \downarrow g \\ \mathcal{P}' & \xleftarrow[\alpha']{} & \mathcal{C}' \end{array}$$

morphism of (weight graded) dg - op / coop. connected. $(\Leftrightarrow \mathcal{P}^{(0)} = \Bbbk\,\mathrm{id}.)$

both are homological alg. proof using S.S. of filtered cpx.

~~functorial~~

<u>functoriality.</u> $\mathcal{P}\circ_\alpha\mathcal{C} \xrightarrow{f\circ g} \mathcal{P}'\circ_{\alpha'}\mathcal{C}'$, $\quad \mathcal{C}\circ_\alpha\mathcal{P} \xrightarrow{g\circ f} \mathcal{C}'\circ_{\alpha'}\mathcal{P}'$ chain maps

$\left.\begin{array}{l}(f, g, f\circ g) \\ (f, g, g\circ f)\end{array}\right]$ satisfies "2/3 property"

fundamental theorem.

$\mathcal{P}, \mathcal{C}$: wgd. (co)operad. TFAE :$\rightsquigarrow$ Def α : Koszul

① $\mathcal{C} \circ_\alpha \mathcal{P}$: acyclic

② $\mathcal{P} \circ_\alpha \mathcal{C}$: acyclic

③. $\mathcal{C} \xrightarrow{f_\alpha} B\mathcal{P}$: q-iso.

④ $\Omega\mathcal{C} \xrightarrow{g_\alpha} \mathcal{P}$: q-iso.

coop | op.

$$\begin{tikzcd} & B\mathcal{P} \arrow[dr, "\pi"] & \\ \mathcal{C} \arrow[ur, "f_\alpha"] \arrow[rr, "\alpha"] \arrow[dr, "\iota"'] & & \mathcal{P} \\ & \Omega\mathcal{C} \arrow[ur, "g_\alpha"'] & \end{tikzcd}$$

prf.

$$\begin{tikzcd} & ③ \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ ① \arrow[dr, leftrightarrow] & & ② \arrow[dl, leftrightarrow] \\ & ④ & \end{tikzcd}$$

$$\begin{array}{ll} \mathcal{C} \circ_\alpha \mathcal{P} & \\ \downarrow f_\alpha \circ \mathrm{id} & : \text{q-iso} \iff f_\alpha : \text{q-iso.} \\ B\mathcal{P} \circ_\alpha \mathcal{P} & : \overset{\text{Lem②}}{\text{acyclic.}} \end{array}$$

$\square$

Cor. ι, α : Koszul.

unit & counit. $\left[\begin{array}{l} \Omega B\mathcal{P} \xrightarrow{\sim} \mathcal{P} \\ \mathcal{C} \xrightarrow[\sim]{} B\Omega\mathcal{C} \end{array} \right.$ bar-cobar resol.

Ω & B preserves quasi-isom. $\boxed{\text{Quillen equiv?}}$

③ Koszul duality.

$E \in {}^{gr}\mathrm{Mod}^{\Sigma^{op}}$.

Def. Quadratic data (E, R). $R \subset \mathcal{T}(E)^{(2)}$. graded sub Σ-mod.

$\forall \mathcal{P}$; 0

operad: $R \hookrightarrow \mathcal{T}(E) \twoheadrightarrow \mathcal{P}(E,R)$. operadic quotient.

cooperad: $\mathcal{C}(E,R) \rightarrowtail \mathcal{T}^c(E) \twoheadrightarrow \mathcal{T}^c(E)^{(2)}/R$. $\boxed{\in \mathrm{grOp}.}$

$\forall \mathcal{C}$

e.g. (nonsym case). $\left\{ \begin{array}{l} E(2) = (\mu = Y), \ \deg 0. \\ E(n) = 0 \quad b(n). \end{array} \right.$ $R = Y - Y$

$\rightsquigarrow \mathcal{T}(E) = k[PBT(n)] \rightsquigarrow As = \mathcal{P}(E,R)$

Def. Koszul dual cooperad of $\mathcal{P}=\mathcal{P}(E,R)$ is

$$\mathcal{P}^{¡} := \mathcal{C}(sE, s^2R).$$

(Koszul dual operad

$$\mathcal{P}^! := \left(\mathcal{S}^c \underset{H}{\otimes} \mathcal{P}^{¡}\right)^* = \mathcal{P}\left(s^{-1}\mathcal{S}^{-1} \underset{H}{\otimes} E^*, R^{\perp}\right)$$

∃ explicit quad repn

$(\mathcal{P}^!)^! = \mathcal{P}$ if E is fin.dim in each arity.

$As^! = As$, $Com^! = Lie$)

~~i.e. $\mathcal{P}^{!}(n) := \mathrm{Hom}($~~ $H_0(B\mathcal{P})$ $= \mathcal{S}^c(s\mathcal{P}(E,R))$.

$$\begin{tikzcd}
 & B(\mathcal{P}(E,R)) \arrow[dr, "\pi"] & \\
\mathcal{C}(sE, s^2R) \arrow[ur] \arrow[rr, "K"] \arrow[d] & & \mathcal{P}(E,R) \\
sE \arrow[r, "s^{-1}"] & E \arrow[ur] &
\end{tikzcd}$$

: twisting mor. $\boxed{K \star K = 0}$

($\Leftrightarrow \Omega\mathcal{P}^{¡} \xrightarrow{\sim} H_0(\Omega\mathcal{P}^{¡})$)
$\Leftrightarrow B\mathcal{P} \xrightarrow{\sim} H_0(B\mathcal{P})$)

Def. $\mathcal{P}$ is Koszul when $K: \mathcal{P}^{¡} \to \mathcal{P}$ is Koszul $\Leftrightarrow \Omega\mathcal{P}^{¡} \xrightarrow{\sim} \mathcal{P} \Leftrightarrow \mathcal{P}^{¡} \xrightarrow{\sim} B\mathcal{P}$.

$\boxed{\text{Koszul resol}}$ q.is of dg op.

$\mathcal{P}_\infty := \Omega\mathcal{P}^{¡}$ e.g. As, Com, Lie, ...

by concrete calculation

example. $A_\infty = (\mathbb{T}(\text{Y}, \Psi, \dots), d) \xrightarrow{\sim} As$

$= As_\infty$ → $\Omega B As$ $\boxed{\text{cubical decomp. W-construction}}$

$$H^\bullet_{sing}(X) \underset{p}{\overset{i}{\rightleftarrows}} C^\bullet_{sing}(X) \circlearrowright h.$$

~~$X \in Top \rightsquigarrow C^{sing}_\bullet(X) \in As$-alg~~

A_∞-alg. As-alg.

μ_3 (classical) Massey product

⇝ detect nontriviality of the complement of Borromean rings