The diagonal of the associahedra

Naruki Masuda JHU

References

[MTTV 19] The diagonal of the associahedra
(Masuda – Thomas – Tonks – Vallette)

improved by

[LA 21] The diagonal of the operahedra
(Laplante - Anfossi)

previous works

[SU 04] Diagonals on the permutahedra, multiplihedra, and associahedra.
(Saneblidze – Umble)

[MS 06] Associahedra, cellular W - construction and products of A_∞ - algebras
(Markl - Shnider)

② Associahedra

$X \in \mathrm{Top}_* \rightsquigarrow \Omega X$ associative alg upto higher coherent homotopies

encoded by an operad map

$$\mu_n : K_n \longrightarrow \underline{\mathrm{Map}}(\Omega X^n, \Omega X)$$

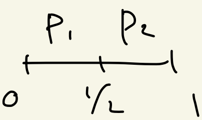
K_n: an $(n-2)$-dim polytope

$$K_0 = * \longrightarrow \mathrm{const}_*$$

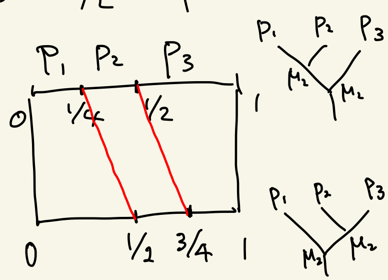
$$K_1 = * \longrightarrow \mathrm{id}_{\Omega X}$$

$$K_2 = * \longrightarrow \Omega X^2 \to \Omega X$$

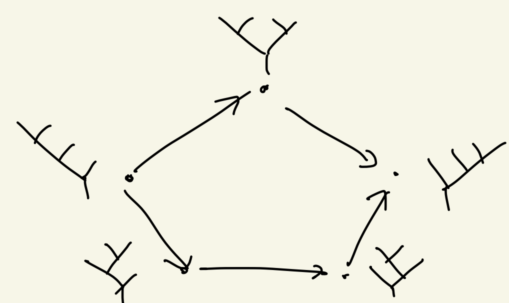
$$(p_1, p_2) \longmapsto$$



$K_3 =$  $\longrightarrow (p_1, p_2, p_3) \longmapsto$



$K_4 =$

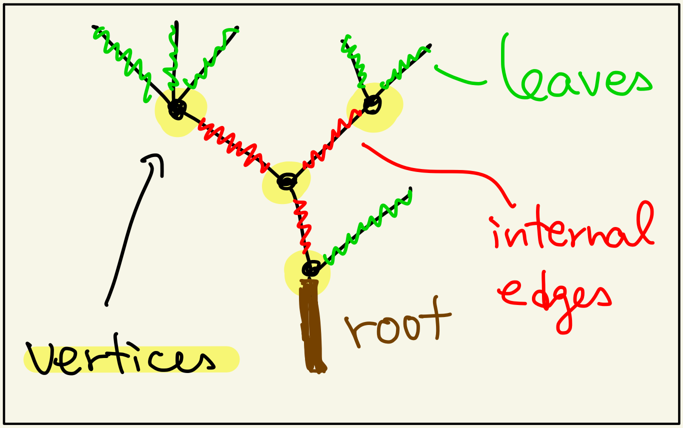


Generally,

$$\mathcal{L}(K_n) \cong PT_n = \left\{ \begin{array}{l} \text{planar (reduced, rooted)} \\ \text{trees with } n \text{ leaves} \end{array} \right\}$$

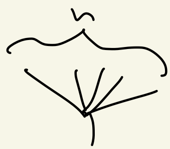
$\uparrow$ face lattice of a polytope

$\uparrow$ lattice by $s \le t \in PT_n \iff s \rightsquigarrow t$ collapsing internal edge(s)



codim of a face $\longleftrightarrow$ # of internal edges

top face of K_n $\longleftrightarrow$ C_n :



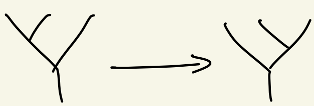
Corolla

codim 1 face $\longleftrightarrow$ $C_k \circ_i C_\ell$

$$sk_0 K_n \cong PBT_n = \{ \text{— '' —}, \text{ binary} \}$$

<u>Tamari lattice</u> $s \xrightarrow{(\le)} t \in PBT_n$:

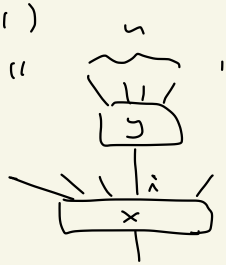
generated by



③ A_∞-structure

<u>Def</u> (non-sym) operad $\mathcal{O}$ in a MC V is

- $\mathcal{O}(n) \in V$ for $n \geq 0$
- $1_{\mathcal{O}} : I_V \longrightarrow \mathcal{O}(1)$
- $\circ_i : \mathcal{O}(m) \otimes \mathcal{O}(n) \longrightarrow \mathcal{O}(m+n-1)$

  $x \otimes y \longmapsto x \circ_i y$ "  "

for $m, n \geq 0$, $1 \leq i \leq m$

that are unital & associative.

<u>ex</u>

- $\mathrm{End}_X(n) = \underline{\mathrm{Hom}}(X^{\otimes n}, X)$ for $X \in V$, V closed
- $(\{PT_n\}, \circ_i, |: \text{trivial tree}) \in \mathrm{Op}((gr)\,\mathrm{Set})$

  $\cup$

  $\{PBT_n\}$ sub-operad

models of "A_∞-operad"

- $\exists$ operad str. on $\{K_n\}$ s.t.

  $\circ_i : K_n \times K_m \xrightarrow{\cong} (\text{face } C_n \circ_i C_m)$

  $\subset K_{n+m-1}$

  CW chain cplx

- $\{C_*(K_n)\} \in \mathrm{dgMod}$

<u>Def</u> $\mathcal{O}$-algebra str. on X $\quad (\mathcal{O} \in \mathrm{Op}(V), X \in V)$

is an operad mor $\mathcal{O} \longrightarrow \mathrm{End}_X$

Thm (Stasheff) Y: connected $\Rightarrow$ TFAE

(i) $\exists X$, $\Omega X \simeq Y$

(ii) Y admits an A_∞-structure, i.e.

$\exists$ mor of operads $\{ K_n \xrightarrow[\mu_n]{} \underline{Map}(Y^n, Y) \}$

ex An A_∞-algebra $= \{ C_*(K_n) \}$-alg in dgMod

unpacking: $\bullet$ $A_\bullet \in$ dgMod

$$\begin{array}{ccc} \bullet\, C_*(K_n) & \longrightarrow & \underline{Hom}(A_\bullet^{\otimes n}, A_\bullet) \\ \cup & & \cup \\ c_n : & \longmapsto & d_n \end{array}$$

(in deg $n-2$)

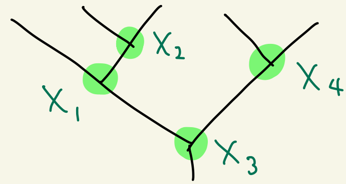
- $d_1 : A_\bullet \longrightarrow A_{\bullet -1}$ differential
- $d_2 : A_\bullet^{\otimes 2} \longrightarrow A_\bullet$ multiplication
- $d_3 : A_\bullet^{\otimes 3} \longrightarrow A_{\bullet+1}$ homotopy

  $d_2 \circ (d_2 \otimes id) \sim d_2 (id \otimes d_2)$

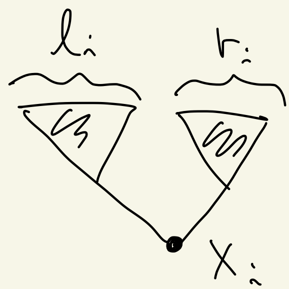
- $d_{>3}$: higher homotopies

③ Loday realization of K_n

$t \in PBT_n$, label vertices from left to right

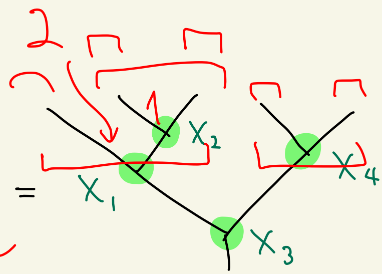


for a vertex



assign $x_i = l_i \cdot r_i$

$t =$



$\in PBT_5$

$$\Rightarrow M(t) = \underbrace{(2, 1, 6, 1)}_{\Sigma = 10} \in \mathbb{R}^4$$

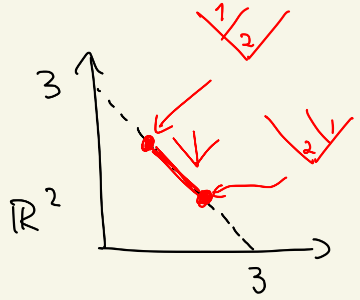
<u>Thm</u>

$$K_n = \mathrm{conv}\{ M(t) \in \mathbb{R}^{n-1} \mid t \in PBT_n \}$$

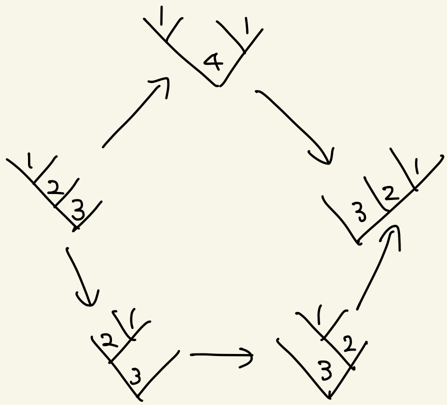
realizes associahedron of dim$(n-2)$

<u>Note</u> $$\underbrace{K_n}_{(n-2)\text{-dim}} \subset \underbrace{\left\{ \sum_{i=1}^{n-1} x_i = \binom{n}{2} \right\}}_{\text{hyperplane}} \subset \mathbb{R}^{n-1}$$

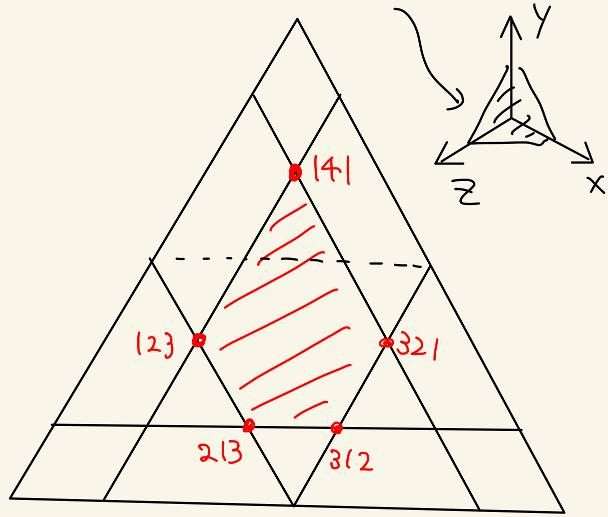
__ex__ K_3



K_4



$x+y+z=6$



__proof idea__

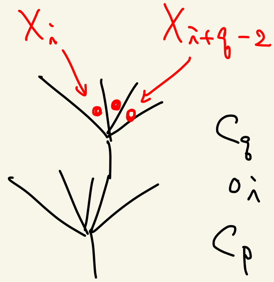
$t \in PBT_{p+q-1}$ admits a decomp $t = \underset{PBT_p}{t'} \circ_i \underset{PBT_q}{t''}$

iff $M(t) = (x_1, \dots, x_{p+q-2})$ satisfies

$$x_i + \dots + x_{i+q-2} = \binom{q}{2}.$$

In general we have $\geq$.

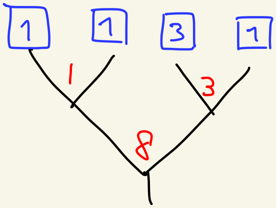
$\rightsquigarrow$ The codim 1 face $C_p \circ_i C_q$ is cut out by $x_i + \dots + x_{i+q-2} \geq \binom{q}{2}$.

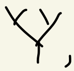


$\blacksquare$

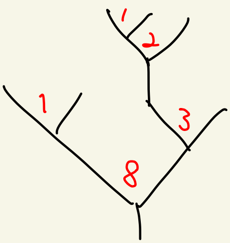
"weighted" version works equally well

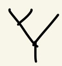
e.g.

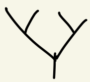


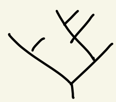
$\rightsquigarrow$ $M($$, (1,1,3,1)) = (1, 8, 3)$

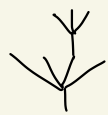
$\rightsquigarrow K_{1,1,3,1} := \mathrm{conv}\,\{ M(t, (1,1,3,1)) \mid t \in PBT_n \}$



$M($$)$

$M($$, (1,1,3,1))$

$M($$)$

face corresponding to $C_4 \circ_3 C_3$  of K_6

is (up to permutation of coord) $K_{(1,1,3,1)} \times K_3$

Generally (face $C_p \circ_i C_q$ of K_{p+q-1})

|| perm of coord

$$K_{\underbrace{(1,\dots,q,\dots,1)}_{p}} \times K_p$$

③ Diagonal Problem

Q Define $\mathcal{O}$-alg str. on $X \otimes Y$ for $\mathcal{O}$-alg X, Y

A. Need operadic diagonal

$$\begin{tikzcd}
\mathcal{O}(n) \arrow[r, dashed] \arrow[d, "\Delta"'] & \underline{\mathrm{Hom}}((X\otimes Y)^{\otimes n}, X\otimes Y) \\
\mathcal{O}(n)\otimes\mathcal{O}(n) \arrow[r] & \underline{\mathrm{Hom}}(X^{\otimes n}, X)\otimes\underline{\mathrm{Hom}}(Y^{\otimes n}, Y) \arrow[u]
\end{tikzcd}$$

Q Δ for A_∞-operad ?

A.1. $A_\infty \xrightarrow{\sim} As$ $\quad (As(n) = * \ \forall n)$

is a cofib repl in $\mathrm{Op}(\mathrm{dgMod})^{\mathrm{proj}}$

$$\rightsquigarrow \begin{tikzcd}
O \arrow[rr] \arrow[d, tail] & & A_\infty \otimes A_\infty \arrow[d, "\sim"] \\
A_\infty \arrow[r, two heads] \arrow[urr, dashed, "\exists"] & As \arrow[r] & As\otimes As
\end{tikzcd}$$

A.2 explicit model of A_∞

$\rightsquigarrow$ want explicit (hopefully computable) Δ

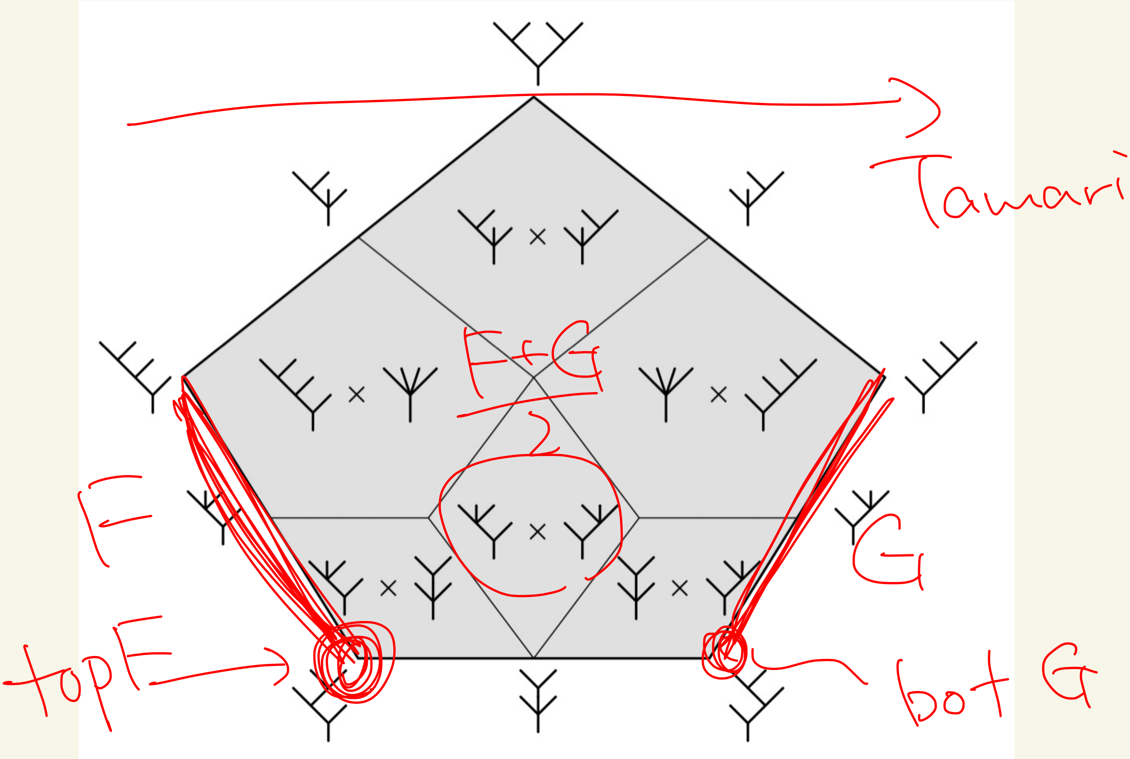
"The magical formula" : $\Delta_n : C_*(K_n) \longrightarrow C_*(K_n)\otimes C_*(K_n)$

$$C_n \longmapsto \sum_{③} t\otimes s$$

[SU 04] [MS 06]

named by Loday

$\left[\begin{array}{l} t, s \in PT_n,\ |t|+|s| = n-2 \\ t \leq s \text{ w.r.t. Tamari order} \end{array}\right]$



A.3 I want to understand this geometrically

Problem(?) Provide conti maps $K_n \xrightarrow{\Delta_n} K_n \times K_n$ s.t.

(1)-(i) a face $\longrightarrow$ $\coprod$ face $\times$ face

(ii) vertex $v \longmapsto (v, v)$

(iii) $\Delta_n \simeq (z \mapsto (z, z))$

(2) Competible w/ the operad str.

and identify the formula

$$\operatorname{Im} \Delta_n = \coprod_{??} F \times G$$

Q2

Q1

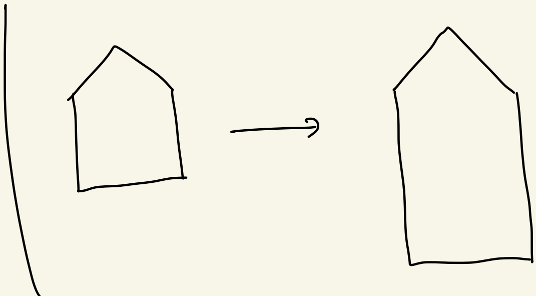
Q3

(iii) The category of polytopal subdivision

What's an appropriate ambient category?

- We will exploit linear algebraic structure of polytopes (CW cplx w/ CW str. is not rigid enough)
- "maps of polytopes" in the literature is affine. too restrictive

  e.g. No affine maps of pentagons



but these are normally equiv

want to add face-respecting homeomorphisms

- polytopal subdivision should be taken in account.

Def A polytopal complex is a finite collection $\mathscr{C}$ of polytopes of $\mathbb{R}^n$ s.t.
$$\begin{cases} \bullet\ F \underset{\text{face}}{\subset} P \in \mathscr{C} \Rightarrow F \in \mathscr{C} \\ \bullet\ P, Q \in \mathscr{C} \Rightarrow P \cap Q : \text{face of } P \,\&\, Q \end{cases}$$

$$|\mathscr{C}| = \bigcup_{P \in \mathscr{C}} P, \quad \mathcal{L}(\mathscr{C}) = \bigcup_{P \in \mathscr{C}} \mathcal{L}(P)$$

Def A poly subdiv. of a polytope $P \in \mathbb{R}^n$ is a poly. cplx $\mathcal{C}$ s.t. $P = |\mathcal{C}|$

Def Poly: **obj** ($n \geq 0$, polytope $P \subset \mathbb{R}^n$)

mor
$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$
conti map s.t.

(i) $P \xrightarrow[\text{homeo}]{} f(P) = |\mathcal{D}| \subset Q$ for some subcpx $\mathcal{D} \subset \mathcal{L}(Q)$

(ii) $f^{-1}(\mathcal{D}) \subset \mathbb{R}^n$ is a polytopal complex

($\Rightarrow$ poly sub of P)

Rmk • isom in Poly is a face-respecting homeo.

• Poly is sym mon by

$$(P \subset \mathbb{R}^n) \times (Q \subset \mathbb{R}^m) := (P \times Q \subset \mathbb{R}^{n+m})$$

$$\begin{array}{c} \text{Poly} \\ \downarrow \\ \text{CW} \\ \downarrow C_* \\ \text{dgMod} \end{array}$$

Sym. mon $\Rightarrow$ Op(Poly) $\longrightarrow$ dgOp

Q (Reformulated)

Q1 Define a family of realizations $\{K_n\}$, give a Poly-op str.

Q2 Give a mor $\Delta_n : K_n \longrightarrow K_n \times K_n$ of Poly-operads

Q3 Describe the subcomplex $\mathrm{Im}\,\Delta_n$ ($\Rightarrow$ magical formula)

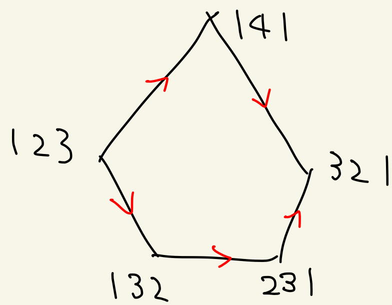
③ Construction of Poly-diagonal

Setting $(P \subset \mathbb{R}^n) \in \mathrm{Poly}$,

$\vec{v} \in (\mathbb{R}^n)^\vee$: a choice of "positive" direction

e.g. • $P = \overset{0}{\bullet}\overset{<}{\longrightarrow}\overset{1}{\bullet} \subset \mathbb{R}^1$, $\vec{v} = (1)$

• $\vec{v} = (v_1, \dots, v_{n-1}) \in (\mathbb{R}^n)^\vee$ induces Tamari order on (weighted) Loday real. K_n iff $v_1 > v_2 > \cdots > v_{n-1}$



$\subset \mathbb{R}^3$

$\longrightarrow \vec{v} = (2, 0, -2)$

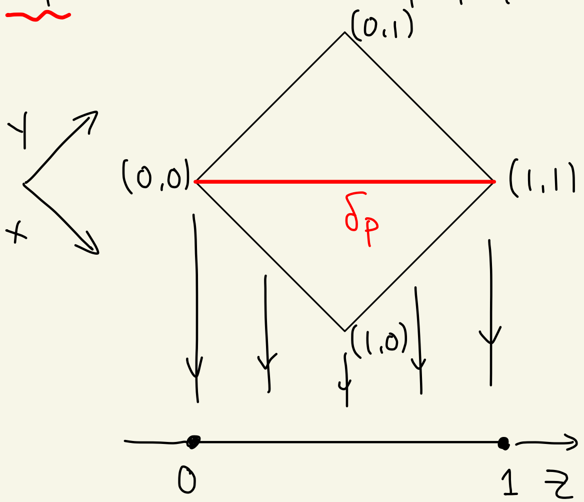
Want $\Delta_{P,\vec{v}} : P \longrightarrow P \times P$ in Poly

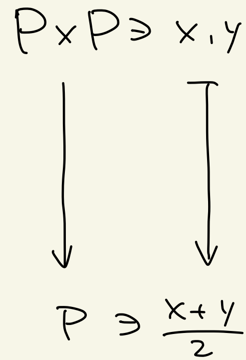
(1) $\Delta_{P,\vec{v}} \simeq \delta_P : z \longmapsto (z, z)$

(2) $\Delta_{P,\vec{v}} = \delta_P$ on vertices

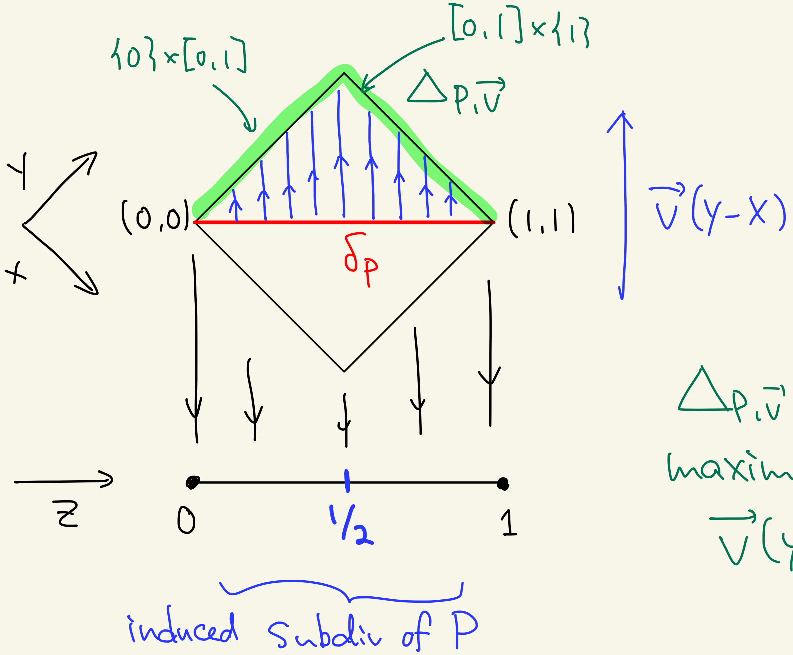
Idea (Motivated by the theory of fiber polytope)

δ_P : section of a polytope bundle



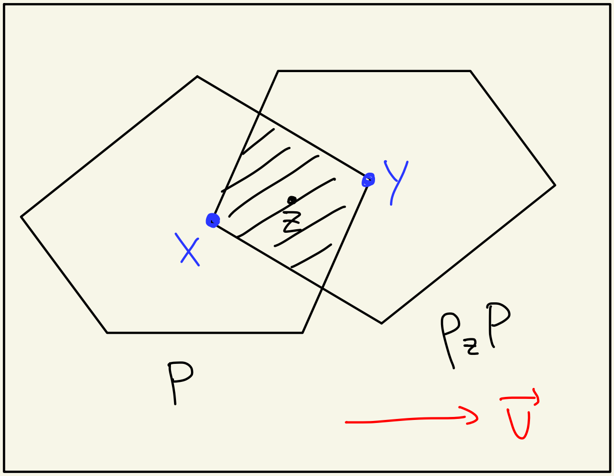


Deform fiberwise to make it polytopal



$\Delta_{P,\vec{v}}$ is a section maximizing $\vec{v}(y-x)$

For fixed $z \in P$, $\beta^{-1}(z) = \{(x,y) \mid \frac{x+y}{2} = z\}$.
both x, y moves in $P \cap (\underbrace{2z - P}_{\substack{\| \\ \rho_z P}})$ refl at z



$\vec{v}(y-x)$ is maximized
$\Leftrightarrow \vec{v}(y)$ is max
$\Leftrightarrow \vec{v}(x)$ is min

$$\rightsquigarrow \Delta_{P,\vec{v}} : P \longrightarrow P \times P$$
$$z \longmapsto (\mathrm{bot}_{\vec{v}}(P \cap \rho_z P), \mathrm{top}_{\vec{v}}(P \cap \rho_z P))$$
(whenever it's well-defined)

<u>Obs</u> • $\Delta_{P,\vec{v}}$ is a mor in Poly satisfying (1)(2)

• Works for $\vec{v}$ in general pos. : enough to exclude

$$\mathcal{H}_P := \{ e^\perp \subset (\mathbb{R}^n)^\vee \mid \begin{array}{l} z \in P, \\ e: \text{an edge of } P \cap \rho_z P \end{array} \}$$

$\not\ni \vec{v} \Rightarrow$ we say $(P, \vec{v})$: positively oriented

$\Delta_{P,\vec{v}}$ only depends on the chamber of $\vec{v} \in ((\mathbb{R}^n)^\vee \setminus \mathcal{H}_P)$.

• $F \subset P$ face $\Rightarrow \Delta_{F,\vec{v}} = \Delta_{P,\vec{v}}|_F$

Ans. to Q1 & Q2

$$\begin{cases} K_n \subset \mathbb{R}^{n-1} \text{ Loday realization,} \\ \vec{v}_n \text{ of decreasing coord } (\Longleftrightarrow \text{in "Tamari chamber"}) \end{cases}$$

$$\Delta_n := \Delta_{K_n, \vec{v}_n} : K_n \longrightarrow K_n \times K_n$$

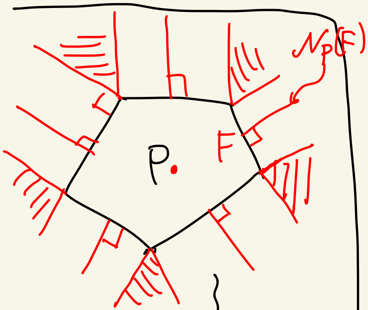
<u>Operad structure?</u>

recall

$$\begin{bmatrix} \text{Codim 1 face} \\ C_p \underset{i}{\circ} C_q \text{ of } K_n \end{bmatrix} \underset{\substack{\uparrow \\ \text{up to shuffle of coord}}}{=} K_{\underbrace{(1,\dots,\overset{i\text{th}}{q},\dots,1)}_{p}} \times K_q$$

$$\begin{array}{ccc} \cap & & \uparrow \text{tr} \times \text{id} \ ?? \\ K_n & \xleftarrow{\ \circ_i\ } & K_p \times K_q \end{array}$$

<u>Lem</u> $P, Q \subset \mathbb{R}^n$: normally equiv ($\mathcal{N}(P) = \mathcal{N}(Q)$).



normal fan

$\mathcal{N}(P)$

$$\Downarrow$$

$$\Phi : \mathcal{L}(P) \cong \mathcal{L}(Q)$$

pos ori by $\vec{v} \in (\mathbb{R}^n)^\vee$

$\Longrightarrow \Phi \times \Phi$ induce bijection

$$\mathcal{L}(\mathrm{Im}\,\Delta_{P,\vec{v}}) \cong \mathcal{L}(\mathrm{Im}\,\Delta_{Q,\vec{v}})$$

<u>Proof</u> $F \times G \subset \mathrm{Im}\,\Delta_{P,\vec{v}}$

$$\Longleftrightarrow \vec{v}^{-1}(\mathbb{R}_{>0}) \cap -\mathcal{N}_P(F)^* \cap \mathcal{N}_P(G)^* = \phi$$

$\uparrow$ Use polar cone thm

Lemma $(P,\vec{v}), (Q,\vec{w})$ pos. ori, $\Phi : \mathcal{L}(P) \xrightarrow{\cong} \mathcal{L}(Q)$

inducing bijection $\mathcal{L}(\mathrm{Im}\,\Delta_{P,\vec{v}}) \xrightarrow[\Phi\times\Phi]{} \mathcal{L}(\mathrm{Im}\,\Delta_{Q,\vec{w}})$

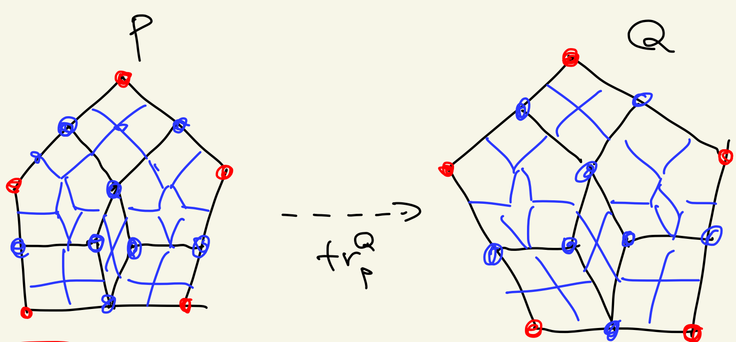
$\Rightarrow \exists!$ conti $tr_P^Q : P \to Q$ s.t.

(1) extends $\Phi : sk_0 P \to sk_0 Q$

(2) commute w/ Δ

Moreover, tr_P^Q : isom in Poly.

idea



$\rightsquigarrow \exists!\ tr : K_p \to K_{\underbrace{(1,\dots,q,\dots,1)}_{p}}$ comm. w/ Δ

$\rightsquigarrow \circ_i : K_p \times K_q \xrightarrow{tr\times id} K_{(1,\dots,q,\dots,1)} \times K_q \overset{\text{shuffle}}{\hookrightarrow} K_{p+q-1}$; comm. w/ Δ

$(\vec{v_p}, \vec{v_q}) \xleftarrow{\ res\ } \vec{V}_{p+q-1}$

associativity is automatic!

$$\begin{array}{ccc} & K_{p+q-1}\times K_r & \\ \nearrow & & \searrow \\ K_p\times K_q\times K_r & & K_{p+q+r-2} \\ \searrow & & \nearrow \\ & K_p\times K_{q+r-1} & \end{array}$$

both comm. w/ Δ

$\Downarrow$

must agree

(11) Ans to Q3

Prop [LA21, Prop 1.15]

$(P, \vec{v})$ s.t. $\forall$ edge of $P \not\perp \vec{v}$

($\Rightarrow sk_0 P$: poset)

$\rightsquigarrow \operatorname{Im} \Delta_P \underset{(\star)}{\subset} \bigcup_{\substack{\mathrm{top} F \\ \leq \mathrm{bot} G}} F \times G$

idea more systematic use of normal fans.

(Rmk In [MTTV19] it took ad-hoc 3 pages argument)

$(P, \vec{v})$ is "magical" if $(\star)$ is equal.

Thm $\exists!$ chamber of $\vec{v}$ inducing the usual order on on vertices of Δ^n, $\square^n$, K_n, all are magical.

Proof $\square^n$: easy, Δ^n: exercise

K_n: Use $PBT_n \twoheadrightarrow \{0,1\}^n$, some combinatorics of trees.

example (AW map)

$$\begin{tikzcd} \Delta^n \arrow[r] \arrow[dr, "\cong"'] & \Delta^n \times \Delta^n \\ & \bigcup_{0\le i\le n} \Delta^{\{0,\dots,i\}} \times \Delta^{\{i,\dots,n\}} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

top, bot

$$A, B \in Ab^{\Delta^{op}} \quad \left(\xrightarrow{C_*} dgMod_{\mathbb{Z}} \right)$$

$$\rightsquigarrow C_*(A\otimes B) \longrightarrow (C_* A)\otimes(C_* B)$$

$$\begin{array}{ccc} C_n(A\otimes B) & \longrightarrow & \bigoplus_{i+j=n} (C_i A)\otimes(C_j B) \\ \cup & & \\ a\otimes b & & \end{array}$$

$$\begin{tikzcd} \mathbb{Z}(\Delta^n\times\Delta^n) \arrow[r, phantom, "="] & \mathbb{Z}\Delta^n \otimes \mathbb{Z}\Delta^n \arrow[r, "a\otimes b"] & A\otimes B \\ \mathbb{Z}\Delta^n \arrow[u, "\Delta"] \arrow[r, phantom, "="] & \mathbb{Z}\Delta^n \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$AW(a\otimes b) = \sum_{i+j=n} a|_{\Delta^{\{0,\dots,i\}}} \otimes b|_{\Delta^{\{i,\dots,n\}}}$$