References:

1. Bhatt - Eilenberg lectures
2. Kedlaya - Notes on Prismatic cohomology

Setup:

- A, I fixed bdd prism, $\bar{A} := A/I$
- R is p-completely smooth $\bar{A}$-algebra, i.e.
  - R is p-complete
  - $R \otimes^L_{\bar{A}} \bar{A}/p$ is conc. in deg 0, where it is smooth over $\bar{A}/p$

§0. Warmup / Recall (Prismatic complex, HT complex, Prismatic coh)

We have seen $(R/A)_{\Delta}^{opp}$, the naive prismatic site with the "indiscrete" or "chaotic" topology, s.t. all presheaves are sheaves.

Obj:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A/I \arrow[r] & R \arrow[r] & B/IB \end{tikzcd}$$

where (B, IB) is a prism

Mor: Obvious

$$\mathcal{O}_{\Delta}(R \longrightarrow B/IB \longleftarrow B) = B$$
$$\bar{\mathcal{O}}_{\Delta}(R \longrightarrow B/IB \longleftarrow B) = B/IB$$
$$\bar{\mathcal{O}}_{\Delta} \simeq \mathcal{O}_{\Delta}/I\mathcal{O}_{\Delta}$$

$(R/A)_{\Delta}$ is Equipped with a weakly final object, s.t. Cech-Alexander complex computes cohomology of $\mathcal{O}_{\Delta}$ and $\bar{\mathcal{O}}_{\Delta}$

$$\Delta_{R/A} := R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$$

"Prismatic complex"
(p, I)-complete
comm. alg object in $D(A)$
w/ Frobenius action

$$\bar{\Delta}_{R/A} := R\Gamma((R/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta})$$

"Hodge-Tate complex"
p-complete in $D(R)$

$$\Delta_{R/A} \otimes^L_A A/I \simeq \bar{\Delta}_{R/A}$$

E.g. $R = A/I$.

Final object ✓
As presheaves are sheaves,

$$\Gamma(\mathcal{O}_{\Delta}) = A$$
$$\Gamma(\bar{\mathcal{O}}_{\Delta}) = A/I$$

$$R^i\Gamma(\mathcal{O}_{\Delta}) = 0 \quad \text{for } i > 0$$
$$R^i\Gamma(\bar{\mathcal{O}}_{\Delta}) = 0$$

Hodge-Tate comparison thm

Relation between $\overline{\Delta}_{R/A}$ and differential forms on R relative to A/I

Agenda :

1) Describe the H-T comparison map

$$\eta_R^* : (\hat{\Omega}^*_{R/\bar{A}}, d_{dR}) \longrightarrow (H^*(\overline{\Delta}_{R/A})\{*\}, \beta_I)$$

State the thm: η_R^* is an isomorphism.

2) Say something about the components of the proof when (A, I) is crystalline, and the relation to crystalline coh.

Maybe preview Cartier isom.

HT comparison

§ 1.1 : Graded comm rgs :

Let $E^\cdot$ be a graded rg. It is graded comm if

$$ab = (-1)^{mn} ba \qquad (a \in E^n, b \in E^m)$$

E.g. $\bigoplus_{n \geq 0} H^n(K^\cdot)$ for $K^\cdot$ a comm alg object in $D(A)$ where $A \in \text{Ring}$

Differential graded algebra :

Let A be a comm rg.

Dga's are chain complexes $(E^\cdot, d^\cdot)$ where

- E^* is a graded A-algebra (so $E^n \otimes_B E^m \xrightarrow{\cdot} E^{n+m}$)
- $d^*: E^* \longrightarrow E^{*+1}$ satisfies the signed Leibniz rule.

$$d^{n+m}(\underset{\in E^n}{a}\underset{\in E^m}{b}) = d^n(a)b + (-1)^n a\, d^m(b)$$

$(E^\cdot, d^\cdot)$ is "commutative" if $E^\cdot$ is graded commutative.

"strictly commutative" if above + $a^2 = 0$ for a in odd degree

(Equivalent if $E^\cdot$ is 2-t.f..)

E.g. de Rham complex

(strictly) commutative dga / A $\xrightarrow{\text{Forget}}$ A - algebra

admits a left adjt $\Omega^\bullet_{B/A} \longleftrightarrow B$

Pf:

$$\begin{array}{c} B \\ \downarrow \eta \\ E^0 \end{array} \rightsquigarrow \begin{tikzcd} B \arrow[r] \arrow[d, "\eta"'] & \Omega^1_{B/A} \arrow[d, dashed, "\exists! \, \eta'"] \\ E^0 \arrow[r, "d"] & E^1 \end{tikzcd}$$

$\rightsquigarrow$ Unique extension by taking wedge products

as long as

$d(\eta x) \cdot d(\eta x) = 0$

$\forall x \in B$

(automatic from comm if no 2-torsion

but else strict commutativity guarantees it, although is too strong)

Continuous de-Rham complex:

$\hat{\Omega}^i_{R/\bar{A}}$ = derived p-completion (as modules) of $\Omega^i_{R/\bar{A}}$

If $E^\bullet$ is derived p-complete then a map

$\Omega^\bullet_{R/\bar{A}} \longrightarrow E^\bullet$ is promoted to $\hat{\Omega}^\bullet_{R/\bar{A}} \longrightarrow E^\bullet$

Fact: R is $\overset{p \text{ here}}{I}$-completely smooth over $\bar{A}$ $\overset{\text{Elkik}}{\Longleftrightarrow}$ R is derived $\overset{p}{I}$-completion of a smooth $\bar{A}$-alg

Kedlaya 6.5.3

$\Rightarrow$ $\hat{\Omega}^\bullet_{R/\bar{A}}$ is finite proj of correct rk.

R° has no extra p torsion $\because$ sm over $\bar{A}$, & $\bar{A}$ is bdd prism

$\therefore$ derived completion = classical completion

Result follows from geom. arguments

§ 1.2 Let $M \in D(A)$

$$\beta^n : H^n(M \otimes^L_A I^n/I^{n+1}) \longrightarrow H^{n+1}(M \otimes^L_A I^{n+1}/I^{n+2})$$

be the connecting hom obtained from the exact sequence

$$0 \to I^{n+1}/I^{n+2} \longrightarrow I^n/I^{n+2} \longrightarrow I^{n+1}/I^{n+2} \to 0$$

Then $\beta^{n+1} \circ \beta^n = 0$

Pf :

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & I^{n+2}/I^{n+3} \arrow[d] & & & \\
0 \arrow[r] & I^{n+1}/I^{n+3} \arrow[r] \arrow[d] & I^n/I^{n+3} \arrow[r] \arrow[d] & I^n/I^{n+1} \arrow[r] \arrow[d, "id"] \arrow[dll, bend right] & 0 \\
0 \arrow[r] & I^{n+1}/I^{n+2} \arrow[r] \arrow[d] \arrow[uu, bend left, "\beta^{n+1}"] & I^n/I^{n+2} \arrow[r] & I^n/I^{n+1} \arrow[r] \arrow[ll, bend left, "\beta^n"] & 0 \\
 & 0 & & &
\end{tikzcd}$$

Take $M = \Delta_{R/A} \in D(A)$

$$\beta^n : H^n(\Delta_{R/A} \otimes^L_A I^n/I^{n+1}) \longrightarrow H^{n+1}(\Delta_{R/A} \otimes^L_A I^{n+1}/I^{n+2})$$

$$H^n(\bar{\Delta}_{R/A}) \otimes I^n/I^{n+1} = H^n(\Delta_{R/A} \otimes^L_A I^n/I^{n+1}), \qquad H^{n+1}(\Delta_{R/A} \otimes^L_A I^{n+1}/I^{n+2}) = H^{n+1}(\bar{\Delta}_{R/A}) \otimes I^{n+1}/I^{n+2}$$

$$H^n(\bar{\Delta}_{R/A}) \otimes I^n/I^{n+1} = H^n(\bar{\Delta}_{R/A}) \otimes (I/I^2)^{\otimes n} = H^n(\bar{\Delta}_{R/A})\{n\}$$

"Breuil Kisin twist"

$\bigoplus_{n \geq 0} H^n(\bar{\Delta}_{R/A})\{n\}$ is a graded comm. $\bar{A}$-algebra

$\{\beta^n\}_n$ make it d.g.a.

In degree 0, admits a map from R

$$\rightsquigarrow \quad \eta : (\hat{\Omega}^\bullet_{R/\bar{A}}, d_{dR}) \longrightarrow (H^\bullet(\bar{\Delta}_{R/A})\{\bullet\}, \beta)$$

Thm: η is an isom.

Cor: $\quad H^i(\bar{\Delta}_{R/A}) \cong \hat{\Omega}^i_{R/\bar{A}}\{-i\}$

Cor: $\quad \bar{\Delta}_{R/A}$ is perfect

§2 : Now $A = \mathbb{Z}_p$, $I = (p)$

$R = \mathbb{F}_p[x_1, \ldots, x_r]$

Idea of proof:

WTS: $\eta : (\hat{\Omega}^{\bullet}_{R/\bar{A}}, d_{dR}) \longrightarrow (H^{\bullet}(\bar{\Delta}_{R/A})\{\bullet\}, \beta)$ is isom.

1) $\phi_{\bar{A}}^{*} H^{\bullet}(\bar{\Delta}_{R/A}) \underset{\text{(A) Last time}}{\cong}$ crystalline coh mod p

$\cong$ crystalline - de Rham comparison (B)

$= H^{\bullet}(\hat{\Omega}_{R/\bar{A}}, d_{dR})$

(RHS)

2) $\phi_{\bar{A}}^{*} \Omega^{\bullet}_{R/\bar{A}} \cong \Omega^{\bullet}_{R^{(1)}/\bar{A}}$

(LHS)

$$\begin{tikzcd} \mathrm{Spec}\, R^{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}\, R \arrow[d] \\ \mathrm{Spec}\, \bar{A} \arrow[r, "\phi"] & \mathrm{Spec}\, \bar{A} \end{tikzcd}$$

3) Use Cartier isom to show that

$\phi_{\bar{A}}^{*} \eta^{i} : \hat{\Omega}^{i}_{R^{(1)}/\bar{A}} \longrightarrow H^{i}(\hat{\Omega}_{R/\bar{A}}, d_{dR})\{i\}$ is an isom

4) Conclude that each η^{i} is an isom

(Here $\phi_{\bar{A}}$ is id, so trivial, but even when A is more general)

(A)

Let $P = \mathbb{Z}_p\{x_1, \dots, x_r\}$

$J = \ker(P \longrightarrow R)$

$\delta^{m}(x_i) \longmapsto 0$

As seen last time, $P\{\frac{J}{p}\}^{\wedge}$ = prismatic envelope of (P, J) is a weakly final obj

$\Delta_{R/A}$ is computed by

$$P\left\{\frac{J}{p}\right\}^\wedge \rightrightarrows P^2\left\{\frac{J^2}{p}\right\}^\wedge \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots$$

where $J^n = \ker(P^{\otimes n} \longrightarrow P \longrightarrow R)$, $P^{\otimes n} = P^n$

& $R\Gamma_{crys}(R/A)$ is computed by

$$D_J(P) \rightrightarrows D_{J^2}(P^2) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots$$

Now, $\mathrm{Spf}\, P \longrightarrow \mathrm{Spf}\, A$ is a cover in the cat. of p-adic formal schemes & $P^\bullet$ is the ass. Čech-Alexander complex

$\Rightarrow$ $A \longrightarrow P^\bullet$ is a htpy equivalence

$\Rightarrow$ $\phi_{P^\bullet}: P^\bullet \longrightarrow P^\bullet$ is a htpy equivalence as ϕ is isom on A

$\Rightarrow$ $\phi: P^\bullet\left\{\frac{J^\bullet}{p}\right\}^\wedge \longrightarrow \phi^*\left(P^\bullet\left\{\frac{J^\bullet}{p}\right\}\right)^\wedge$ gives a htpy equiv

$$\phi^*\left(P^\bullet\left\{\frac{J^\bullet}{p}\right\}\right)^\wedge \cong P^\bullet\left\{\frac{\phi(J^\bullet)}{p}\right\}^\wedge$$

Fact: $$P^2\left\{\frac{\phi(J^n)}{p}\right\}^\wedge = D_{J^n}(P^{\otimes n})$$

(The uncompleted rings are identical in $P^{\otimes n}\left[\frac{1}{p}\right]$)

$$\Delta_{R/A} \xrightarrow{qis} \phi^*\Delta_{R/A} \underset{\text{(Shenrong)}}{\simeq} R\Gamma_{crys}(R/A)$$

(B) Crystalline to deRham comparison: $R\Gamma_{crys}(R/A) \simeq \hat{\Omega}^\bullet_{P/A} \otimes_P D_J(P)$

We have

$$\begin{tikzcd}
0 \arrow[r] & D_J(P) \arrow[r] \arrow[d] & D_{J^2}(P^2) \arrow[r] \arrow[d] & D_{J^3}(P^3) \arrow[r] \arrow[d] & \cdots \\
& D_J(P) \otimes_P \hat{\Omega}^1_{P/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2) \otimes_{P^2} \hat{\Omega}^1_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots & \cdots \\
& D_J(P) \otimes_P \hat{\Omega}^2_{P/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2) \otimes_{P^2} \hat{\Omega}^2_{P^2/\mathbb{Z}_p} \arrow[r] & \cdots \\
& \vdots & &
\end{tikzcd}$$

Turns out, going down first & then $\longrightarrow$ leaves just the coh. of the first column

Going $\longrightarrow$ first only leaves the coh. of the first row

$\Rightarrow$ $R\Gamma_{crys}(R/A)$ is qis to $D_J(P) \otimes_P \hat{\Omega}^\bullet_{P/\mathbb{Z}_p} \overset{qis}{\cong} \hat{\Omega}^{\bullet, dR}_{\mathbb{Z}_p[x_1, \dots, x_r]/\mathbb{Z}_p}$

Mod p : $R\Gamma_{crys}(R/A) \otimes^L_A A/p \simeq \left(\Omega^\bullet_{R/(A/p)}, d_{dR}\right)$ in $D(A/p)$