- thank for invitation
- Overview.
- Motivation ← main.
- tensor product.

Recall $\mathcal{S}$ ∞-cat of spaces. (anima.

$$Sp = \lim(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*) \quad \infty\text{-cat of spectra.}$$

classical.	ho thry	higher cat thry.
Set $(0,0)$ Set $\xrightarrow{\text{free}}$ Ab, Ab $\xrightarrow{\text{forget}}$ Set	Anima. $(\infty,0)$ $\mathcal{S} \xrightarrow{\Sigma^\infty} Sp$, $Sp \xrightarrow{\Omega^\infty} \mathcal{S}$	(weak) ω-cats. (∞,∞). $\omega Cat \xrightarrow{\Sigma^\infty} CatSp$, $CatSp \xrightarrow{\Omega^\infty} \omega Cat$
$\mathbb{N}$ $\quad \mathbb{Z}$	$\widehat{Fin}$ $\quad \mathcal{S}$ stable ho thry	$\widehat{Fin}$ $\quad$ stable cat thry

$$CatSp = \lim(\cdots \to \omega Cat_* \xrightarrow{\Omega} \omega Cat_*) = \lim(\cdots \to CMon(\omega Cat) \to CMon(\omega Cat).)$$

$$(X,x) \longmapsto End_X(x)$$

Can we do any algebra? Semiadditive.

$\mathcal{S} = Cat_{\infty,0}$ $\quad$ ($(\infty,1)$-cat of spaces).

$(n+1)Cat = (nCat)-Cat$ $\quad$ (enriched ∞-cats). ($(\infty,1)$-cat of.)

$(n+1)Cat \rightleftarrows nCat$ (adjunctions $L \dashv I \dashv R$).

def. $\omega Cat := (\cdots \to (n+1)Cat \xrightarrow{R} nCat \to)$

$\hookrightarrow (\omega Cat)-Cat \simeq \omega Cat.$

CatSp

looks very radical.

~~[illegible]~~ Why do we care?

evolution of abstract algebra. progressively radical.

age of set thry. • vector spaces. / $\mathbb{R}$, $\mathbb{C}$, polynomials.

abstract.
"Universal base"

numerical invariants. (# of pts. polynomials). $\boxed{\text{classical AG}}$

• abelian groups / $\mathbb{Z}$, $\otimes$ → comm. rings. → schemes
$\text{CRing} \to \text{Set}$. modern AG

↳ new way of thinking: FOP; better cat properties.
↳ ~~[illegible]~~ new object: arithmetic
(study integral solns).

• invariants: cohomology groups. $\boxed{\text{abelian cat}}$

• moduli problem.

~~age of homotopy thry~~ • homological method ~~[illegible]~~ (derived cats).
triangulated cat :(

indication ~~need for~~ of ho thry • stacks.
automorphisms;

resolve. objects by simple pieces.
Set thry is not enough
$\boxed{\text{ho thry}}$ provides the way.

(cf. $X \simeq \operatorname{colim}_X *$ ho type)

$$\begin{array}{ccc} \text{FOP}: \text{CRing} & \longrightarrow & \text{Set}. \\ \downarrow & & \uparrow \\ \text{CAlg} & \dashrightarrow & \text{Gpd}. \end{array}$$

• dg-alg works well / $\mathbb{Q}$

Universal base

• SCR. works / $\mathbb{Z}$. ~~[illegible]~~.
(in positive degrees)

Spectra / $\mathbb{S}$ $\boxed{\text{(pre) stable } \infty\text{-cat}}$

gp compl. of $(\text{Fin}^{\simeq})$ (BPQ thm). property.

Sp w/ SM str. $\otimes$ $(\wedge)$.

- FOP: $\mathrm{CAlg}(\mathrm{Sp}) \to \mathcal{S}$ — streamlined def thry via cotangent cplx formalism.
- new object: stable ho thry. $\mathbb{S}$, MU, ...

<u>invariants</u>: cohomology, <u>spectra</u>, & six functors.

(motivic, syntomic).

- alg K-thry. $(\mathrm{Sp})\mathrm{Sch} \to \mathrm{Sp}$ — understanding K(scheme) goes through K(der. sch)

(& THH, etc.).

<u>age of higher cats?</u>

indication. nc inv.

$$\begin{array}{ccc} (\mathrm{Sp})\mathrm{Sch} & \longrightarrow & \mathrm{Sp} \\ {\scriptstyle \mathrm{QCoh}}\searrow & & \nearrow{\scriptstyle K} \\ & \mathrm{Pr}^{L,\omega}_{st} & \\ {\scriptstyle \mathrm{Perf}}\searrow & \uparrow\downarrow & \nearrow \\ & \mathrm{Cat}^{perf}_{st} & \end{array}$$

$\underbrace{\mathrm{Perf}}_{\text{invariant of categories}}$

<u>nc inv.</u>

- secondary K-theory & 2-Vb. ($\longleftrightarrow$ TMF?). <u>cf.</u> cat level vs chromatic level.

new <u>invariant</u>: TQFT $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \text{"}n\mathrm{Vect}_{\mathbb{C}}\text{"})$.

- Bord_n. n-cat. obj: $*$
  - mor: 1-dim mfd. (cob between obj.)
  - 2-mor: 2-dim mfd.
  - $\vdots$
  - n-mor: n-mfd.
  - (higher mor are diffeos & isotopies).

sym mon str. $\amalg$.

- $n\mathrm{Vect}_{\mathbb{C}}$ $\otimes$
  - 0-vect: $\mathbb{C}$
  - 1-Vect: $\mathbb{C}$-v.s. $= \mathrm{Vect}_{\mathbb{C}}$
  - 2-Vect: $\mathbb{C}$-lin Cat $= \mathrm{Mod}_{\mathrm{Vect}_{\mathbb{C}}}(\mathrm{Pr}^L)$ ~~[illegible]~~ $\mathrm{Morita}_{\mathbb{C}}$
  - (dg cat/$\mathbb{C}$ w/ ~~[illegible]~~ presentable)

$\mathbb{C}$-v.s. top
$\downarrow$ $\downarrow$
$\mathbb{C}$-alg, bimod, linear maps.
~~[illegible]~~

~~votgoezz~~ e.g. 1-d TQFT.

$\mathrm{Bord}_1 \longrightarrow \mathrm{LinCat}_{\mathbb{C}}$

- $\phi \longmapsto \mathrm{Vect}_{\mathbb{C}}$
- $* \longmapsto \exists\, \mathcal{C}$ (dualizable)
- $\mathrm{id}_\phi \longmapsto \mathrm{Vect}_{\mathbb{C}} \xrightarrow{\mathbb{C}} \mathrm{Vect}_{\mathbb{C}}$ $\quad \mathbb{C} \xrightarrow{\mathbb{C}} \mathbb{C}$
- $S^1 \longmapsto$ $\mathrm{Vect}_{\mathbb{C}} \longrightarrow \mathcal{C}\times\mathcal{C} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ "$\dim \mathcal{C}$"

$\phi \to \ominus \to \phi$

Cob. hyp. (roughly). value on $*$ determines the functor. must be dualizable.

~~closed manifold~~ $M \mapsto$ ~~number $M \in \mathbb{C}$~~

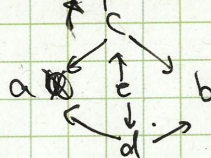
$\mathrm{Vect}_{\mathbb{C}}$ $\mathbb{C}$ $\mathrm{Vect}_{\mathbb{C}}$ $\mathbb{C}$

$x(M) \in \mathbb{C}$

$X \in \mathrm{Sch}_{\mathbb{C}} \rightsquigarrow \mathcal{C} = \mathrm{QCoh}(X)$. "B-model of X".

$\mathrm{Span}(\mathrm{Sch}) \longleftarrow \mathrm{Sch}$

$\mathrm{Bord}_1 \longrightarrow \mathrm{Span}(S^{op}) \xrightarrow{X^{(-)}}$



$\big\downarrow \mathrm{QCoh}$. [GR]

$2\text{-}\mathrm{Vect}_{\mathbb{C}}$ (2-op).

$\left(X \xleftarrow{p} Z \xrightarrow{q} Y \right) \rightsquigarrow \left(\mathrm{QCoh}\, X \xrightarrow{p^*} \mathrm{QCoh}\, Z \xrightarrow{q_*} \mathrm{QCoh}\, Y \right)$

ex. apply to S^1

$$\mathrm{Vect}_{\mathbb{C}} = \mathrm{QCoh}(\mathrm{Spec}(\mathbb{C})) \to \mathrm{QCoh}(X^{S^1}) \to \mathrm{QCoh}(\mathrm{Spec}(\mathbb{C})) = \mathrm{Vect}_{\mathbb{C}}$$

$\mathcal{O}(\mathcal{L}(X)) = HH(X/\mathbb{C})$

Interesting n-dim TQFT from a scheme?

Rem geom. Lang. is expected to upgrade to an equiv. between 4d TQFT.

Stefanich, GR.

$n\mathrm{QCoh} : n\mathrm{Corr}(\mathrm{Sch}) \to n\mathrm{Mod}$.

package

Categorical spectra ~~[illegible]~~ iterated categorification.

$$\mathrm{CatSp} := \lim(\cdots \to \omega\mathrm{Cat}_* \xrightarrow{\Omega} \omega\mathrm{Cat}_*)$$

levelwise gpd compl. $\mathrm{Gr} \downarrow \dashv \uparrow \dashv \downarrow \mathrm{Pic}$.

ex Spectra.

$$\mathrm{Sp} := \lim(\quad S_* \xrightarrow{\Omega} S_* \quad).$$

$$\left(= \lim(\cdots \to \mathrm{Sp}^{cn} \xrightarrow{\Omega} \mathrm{Sp}^{cn})\right)$$

$\mathrm{Sp}^{cn} \cong \mathrm{CMon}^{gp}$

$$\begin{tikzcd} \mathrm{CatSp} \arrow[r, "\Omega^\infty"] & \omega\mathrm{Cat}_* \arrow[l, bend right, dashed, "\Sigma^\infty"'] \arrow[d, shift right, "\mathrm{Free}"'] \\ & \mathrm{CMon}(\omega\mathrm{Cat}) \arrow[u, shift right, "\mathrm{forget}"'] \arrow[ul, shift right, "\mathrm{Fr}, B^\infty"'] \end{tikzcd}$$

Ω^∞

(Semiadditive) CMon-enriched.

cf. Baez-Dolan delooping hyp.

$(M, BM, \cdots) \leftarrow\!\shortmid M \in \mathrm{CMon}(\omega\mathrm{Cat})$

ex suspension spectra for $X \in \omega\mathrm{Cat}_*$.

$$\Sigma^\infty X = B^\infty \mathrm{Free}_{E_\infty}(X).$$

- sub-ex $X = S^0 \rightsquigarrow B^\infty \mathrm{Fin}^{\simeq}$

$\mathbb{Z}$, $\mathbb{N}$, $\mathbb{S}$, $\mathbb{A}$ deeper.

~~[illegible]~~ ex EM spectra. M: comm. monoid $\rightsquigarrow B^\infty M$.

"Brauer" cat sp. R: $\mathbb{E}_\infty$-ring. $\rightsquigarrow \mathrm{Mod}_R(\mathrm{Sp}) \in \mathrm{CAlg}(\mathrm{Pr}^L_{st})$.

$$2\text{-}\mathrm{Mod}_R := \mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^L_{st}). \quad \cdots$$

$$(R = \mathbb{S} \rightsquigarrow n\mathrm{Pr}^L_{st}).$$

$$\begin{gathered} \{n\mathrm{Mod}_R\} \\ \downarrow \mathrm{Pic}. \\ (R^\times, \mathrm{Pic}R, \mathrm{Br}R, \underbrace{\cdots}_{??}). \end{gathered}$$

$\boxed{n\text{-Corr}}$

$\mathcal{C}$: cat w/ fin lim

$\mathrm{Corr}(\mathcal{C})$

$= \{n\mathrm{Corr}(\mathcal{C})\}_n$

universal semiadditive.

• "Super" version (JF-Reutter) e.g. $\underset{\sim}{s}\mathrm{Vect}_{\mathbb{C}}$ for fermions.

can be characterized by $^{\circledast}$"Nullstellensatz-like condition.

↳ $\mathrm{nsVect}_{\mathbb{C}} \xrightarrow[\mathrm{Pic}]{} I_{\mathbb{C}^\times}$ BC dualizing spectrum.

• universal "physical" target for TQFT.

↳ universal ______ inv. TQFT $I_{\mathbb{C}^\times}$

(Freed-Hopkins ansatz).

new objects : ~~Connes-Consani~~

• better approx to $\mathbb{F}_1$? (CC abs geom).

(their homological alg essentially uses $\mathbb{S}$).

Need tensor product! CatSp

recall $\otimes$ on Ab was unique s.t. { Free : Set → Ab strong sym. mon, distributive over colim. }

Similarly for Sp. { $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$ — " — , — " — . }

Rem $\mathcal{M}$: $(\infty$-$)$ monoidal cat. $1_{\mathcal{M}} \xrightarrow{\eta} X$ is idempotent if

$\eta \otimes \mathrm{id}_X$, $\mathrm{id}_X \otimes \eta$: equiv

⟶ • X has a canonical lift in $\mathrm{CAlg}(\mathcal{M})$

• X-mod str. is a property in $\mathcal{M}$: $\mathrm{LMod}_X(\mathcal{M})$ $\overset{\otimes}{\rightleftarrows}$ $\mathcal{M}$ localization

ex $\mathbb{Z} \to \mathbb{Q}$ in Ab. $\mathbb{Q} \otimes \mathbb{Q} \simeq \mathbb{Q}$.

<u>ex</u> $\mathcal{S} \longrightarrow Sp$ in Pr^L. (Rmk Pr^L has sym $\otimes$ making $\mathcal{P} : Cat \to Pr^L$ strong mon, distributes colim.)

$Sp \otimes Sp \simeq Sp$.

$\rightsquigarrow \mathrm{Mod}_{Sp}(Pr^L) \underset{\longrightarrow}{\overset{Sp\otimes -}{\underset{\perp}{\longleftarrow}}} Pr^L$

$\mathrm{Mod}_{Sp}(Pr^L) \simeq Pr^L_{st}$.

unit $= Sp$ (characterizes $Sp^{\otimes}$ as the unit of Pr^L_{st}!).

<u>Q</u> $\omega Cat_* \xrightarrow{\Sigma^{\infty}_{+}} CatSp$ idem?

<u>$\otimes$</u> ~~does not make sense : ambient~~

must be $1_{\mathcal{M}}$ of the ambient $\mathcal{M}$.

$\rightsquigarrow \mathrm{Mod}_{\omega Cat_*}(Pr^L)$? but Σ^{∞}_{+} is not a ωCat-mod map!

☹ $X \wedge \Sigma Y \not\simeq \Sigma(X \wedge Y)$. cat level don't match.

<u>Fix</u> use $\otimes$ on ωCat s.t. $(n\text{-cat}) \otimes (m\text{-cat}) = n{+}m\text{-cat}$.

<u>(lax) Gray $\otimes$</u>. ⚠ partially Conjectural.

<u>ex</u>. $\downarrow \otimes \rightarrow \; = \;$ (square with 2-cell) $\qquad \square^n \otimes \square^m \simeq \square^{n+m}$.

<u>problem</u> only E_1 $\qquad \rightarrow \otimes \downarrow \;\not=\;$ (square with 2-cell)

~~alth~~ although Σ : left $\omega Cat_*^{\otimes}$-mod hom. $\mathrm{LMod}_{\omega Cat}(Pr^L)$ has no mon. str. $\qquad - \otimes \vec{S}^1 = BN$.

<u>Fix</u> consider $\mathrm{BMod}_{\omega Cat}(Pr^L)$. <u>Q</u>: does Σ lift to BMod how?

does $\vec{S}^1$ lift to the center?

A. No. but $\vec{S}^2$ does.

$\rightsquigarrow$ colim$(\xrightarrow{\Sigma^1} \xrightarrow{\Sigma^2} \cdots) = $ CatSp in BMod. still makes sense.

& can show it's idempotent.

up shot: CatSp $\exists!$ $\mathbb{E}_1$-mon. str. s.t. $\omega\mathrm{Cat} \xrightarrow{\Sigma^\infty} \mathrm{CatSp}$ pres. str. mon.

"$\vec{S}$ acts invertibly" ~~⊗~~.

$\Leftrightarrow$ CatSp-mod. str. on $\omega\mathrm{Cat}^{\otimes}_{\otimes}$-bimod

stability what is this?

Vistas. ~~How~~ To ~~do~~ alg. want more commutativity.

idea ~~X⊗~~ exploit $(X \otimes Y)^{op} \simeq Y^{op} \otimes X^{op}$?