SEMINAR ON PRISMS Δ

Overview: Part I: The discussions roughly follow the self-contained paper, *Prisms and prismatic cohomology*, [10], main accompanying reference is [4].

Part II: we discuss some applications (depending on interest)

- Crystalline Galois representations. [9]
- algebraic K-theory and the motivic filtration, [7].
- Prismatizations, [13], [3]. A very nice companying reference is [5].

The seminar program here serves as a brief guide.

Part I:

0.1. **Motivations.** Survey of motivations. [10, 1], see also [5, 1].

(1) Question: How could one understand integral/$\mathbb{F}_p$ coefficient cohomology? [1] The main comparison theorems, [10, Thm. 1.8]. One implication: [6, 1.1, (ii)], [4, 1], for $X \in \mathrm{Sch}^{\mathrm{sm,proj}}_{\mathcal{O}_C}$, "$\mathbb{F}_p$ cohomology of X_C as an obstruction to integration of forms on X_k."

$$\dim_{\mathbb{F}_p} H^i(X_C, \mathbb{F}_p) \leq \dim_k H^i_{\mathrm{dR}}(X_k)$$

This inequality can be strict.

(2) Motivic filtrations, and their applications to K-theory. Let k be any field, , for $X \in \mathrm{Sch}^{\mathrm{sm,proj}}_k$, there is the *motivic filtration*

$$\mathrm{Fil}^{\geq *}K(X) \text{ where } \mathrm{gr}^i K(X) \simeq \mathbb{Z}(i)^{\mathrm{mot}}(X)[2i] \quad i \geq 0$$

The analogous statement for singular affine schemes is open. [7] proves a statement for its close cousin [2] TC. Remark on syntomic cohomology.

(3) Absolute prismatic cohomology, [3].

0.2. **δ-rings.** Define δ-rings, [10, 2]. State universal property and the adjunction

$$\delta\mathrm{CAlg} \underset{W}{\overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}}} \mathrm{CAlg}$$

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[1] Over $\mathbb{C}$ one uses Hodge theory.

[2] Thanks to advances in trace methods, see [12]

The right adjoint is Witt vectors functor. Describe the δ structure on $\mathbb{Z}\{S\}$ denote the free δ ring with generating set S. Sketch equivalence[3]

$$\delta\text{-CAlg}^{p\text{-tf}} \xrightarrow{\simeq} \{(A, \phi) \,:\, A \in \text{CAlg}, \phi \text{ is lift of Frob}\}$$

$$(A, \delta) \mapsto (A, \phi_\delta), \phi_\delta : x \mapsto x^p + p\delta(x)$$

Remark why these are called derivations, [11]. Give examples, [10, 2.6, 2.11]. A non-example, see [10, 2.35]. State the equivalence [10, 2.31].

$$\delta\text{-CAlg}^{p\text{-cpl,perf}} \underset{W(R) \leftarrow R}{\overset{A \mapsto A/p}{\rightleftarrows}} \text{CAlg}^{\heartsuit,\text{perf}}_{\mathbb{F}_p}$$

The proof boils down to [10, 2.28], but give alternative proof as [10, 2.30]. State equivalence [4, 2.2], [10, 3.10]

$$(\star) \qquad \text{Prisms}^{\text{perf}} \xrightarrow{\simeq} \text{Pftd}$$

This will be discussed in 0.3. Omit sections, [10, 2.5, 2.6], this will be discussed in 0.4.

0.3. **Prisms.** We have established a char. 0 theory of perfect char. p rings. This can already be done with Witt vectors - but we extend this to perfectoid rings. Introduce prisms, [10, 3], recall distinguished elements, [10, 2.19]. Discuss its geometric interpretation. Discuss how [10, 2.20] forms prisms. Focus on crystalline and A_{inf} case. Sketch proof of ($\star$).

In [4, IV], Bhatt takes a definition of a ring to be perfectoid if it is of the form A/I for a perfect prism (A, I). Briefly comment on its relation to the intrinsic versions, [6, 3].

0.4. **Prismatic site.** Define the absolute and relative prismatic site for a general ring R, yielding R_Δ and $(R/A)_\Delta$, where (A, I) is a fixed base prism, see also [3, 4]. Define $\Delta_{R/A}, \bar{\Delta}_{R/A}$. Prismatic site as slice topos, [10, 4.3]. Discuss lemma [10, 2.18].

To gain intuition, we will do some computations and compare the construction in [10, 4] with crystalline cohomology, which plays the same game (with Zariski topology). In particular, state [4, VI, 3.2]

$$\text{(Crys)} \qquad \left(\phi_A^* \Delta_{R/A}\right)^\wedge \simeq \Gamma_{\text{Crys}}(R/A)$$

which is in turn used to prove the Hodge-Tate comparison.

[3] In this setting, δ-rings are also referred as *derivations*.

Crystalline cohomology. A summary can be found in [4, VI, 1], [14]. DP[4]-envelope, [8, 3]. Give examples: [16, Ex. 2.2, 2.3], [4, VI, 1.3]. Give DP envelope of $\mathbb{Z}_p\{x\}$. Crystalline site, [16, 3]. State Crystalline-de Rham comparison, [1]: For $P \twoheadrightarrow R$ surjection with kernel J,

$$\Omega^\bullet_{P/A}\widehat{\otimes}_P D_J(P) \simeq R\Gamma_{\mathrm{Crys}}(R/A) \tag{cdR}$$

State the characteristic 0 version: Grothendieck's infinitesimal cohomology. Mention how one obtains F-crystal.

0.5. **Hodge-Tate comparison, I.** State the Hodge Tate comparison, [5, V,3.8]: If $(A,(d))$ is a bounded prisms, $R \in \mathrm{CAlg}^{\mathrm{fm.sm}}_{A/(d)}$

$$\eta^*_R : \left(\Omega^\bullet_{R/(A/d)}, d_{\mathrm{dR}}\right) \xrightarrow{\simeq} H^*(\bar{\Delta}_{R/A}, \beta_d) \tag{HT}$$

link back to 0.1. Our first goal is to construct this map.

The universal property of de Rham complex. See also [2, 3]. State the universal property for the completed de Rham complex, [5, 3.4]. This is the version we use. Introduce Brueil-Kisin twist [10, 3.7] and define Bockstein homomorphism as those induced from the short exact sequence

$$0 \to I^{n+1}\mathcal{O}_\Delta/I^{n+2} \to I^n\mathcal{O}_\Delta/I^{n+2} \to I^n\mathcal{O}_\Delta/I^{n+1} \to 0$$

of $\mathcal{O}_\Delta$-modules in $(R/A)_\Delta$.

If time permits: Discuss how one can compute crystalline cohomology groups using Čech-Alexander complex and the indiscrete topology, see [4, V], [4, VI, 1.4], [15, 07JK]. That is

$$\Gamma(X_{\mathrm{indis}}, \mathcal{F}|_{\mathrm{indisc}}) \simeq \Gamma((X/S)_{\mathrm{Cris}}, \mathcal{F})$$

Set the stage for discussion for the Crystalline case, $(A,(d)) = (\mathbb{Z}_p,(p))$.

0.6. **Hodge-Tate comparison, II.** Our goal is to discuss the proof of (HT) for crystalline prism, $(\mathbb{Z}_p,(p))$, and R is a polynomial algebra over $\mathbb{F}_p$. This case is sufficient, to generalize, one uses Quillen's formalism of non-abelian derived functors, [4, VII]. Split the argument into two parts. Describe how one deduces from characteristic p case :

- Cartier isomorphism, (Cart)
- the Crystalline comparison theorem, (Crys).

[4] Also referred as PD.

Recall the Cartier isomorphism [2, 3.34] and its variation [4, 1.9]
(Cart)

$$\mathrm{Cart} : (\Omega^{\bullet}_{R^{(1)}/(A/p)}, d_{\mathrm{dR}}) \xrightarrow{\cong} H^*(\Omega^{\bullet}_{R/(A/p)}), \beta_p) \simeq H^*(R\Gamma_{\mathrm{crys}}(R/A) \otimes_A R/p)$$

where the last equivalence is crystalline de Rham comparison, (cdR). Prove the crystalline comparison. [4, 3.2].

References

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