(((very) slowly) towards)

Derived Absolute Algebraic Geometry
(Spectral)

② Absolute AG

$\exists$ deep analogy between number fields & function fields :

number fields	function fields
$\mathbb{Z}$	$\mathbb{F}_p[t]$
$\mathbb{Q}$	$\mathbb{F}_p(t)$
$\operatorname{Spec}\mathbb{Z}$	$\operatorname{Spec}\mathbb{F}_p[t] = \mathbb{A}^1_{\mathbb{F}_p}$
"$\overline{\operatorname{Spec}\mathbb{Z}}$" $= \operatorname{Spec}\mathbb{Z} \cup \{\infty\}$	$\mathbb{P}^1_{\mathbb{F}_p}$
absolute value on $\mathbb{Q}$ $/\sim$ $=$ place v $=$	closed point
$\mathbb{Q}_v = \begin{cases} \mathbb{Q}_p & v = p \in \operatorname{Spec}\mathbb{Z} \\ \mathbb{R} & v = \infty \end{cases}$	$(\mathbb{F}_p(t))_v = \begin{cases} \mathbb{F}_q((t-x)) & (v \in \mathbb{A}^1 \leadsto f(t) \text{ irred}) \\ \mathbb{F}_p((\frac{1}{t})) & (v = \infty) \end{cases}$
$\mathbb{Z}_p$	$\mathbb{F}_q[[t-x]]$
product formula $\prod_v \lvert x\rvert_v = 1$ for $x \in \mathbb{Q}$ or $\mathbb{F}_p(t)$	
$\partial_p = \dfrac{(-)^p - (-)}{p}$	$\partial/\partial t$
Riemann ζ	Hasse-Weil ζ

Similarly for $K/\mathbb{Q}$ fin ext (number field) | $K/\mathbb{F}_p(t)$ fin (sep) ext (function field)

For function fields, geometric tools (e.g. Weil cohomology) are available.

$\rightsquigarrow$ Q. Can we understand number fields in a similar way?

Spec $\mathbb{Z}$ is a "curve over a point"
(or Spec $\mathcal{O}_K$)

Obstacles ① $\mathbb{Z}$ is a initial ring, no "coefficient field"
$\rightsquigarrow$ No object is below Spec $\mathbb{Z}$. (with relative dim 1)

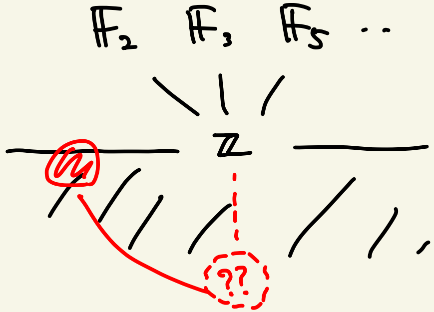
"deeper"

$\Downarrow$

"wider"

② Compactifying Spec $\mathbb{Z} \hookrightarrow \overline{\mathrm{Spec}\,\mathbb{Z}}$ to complete the analogy
projective, or at least proper

"Arakelov compactification" is a subtle issue.



} classical AG (relative / $\mathbb{Z}$)

} absolute AG (over "Spec $\mathbb{F}_1$" "absolute point")

"$\mathbb{F}_q$, $q \to 1$"

Approach Generalize the notion of algebra(ic geometry)

Proposed Generalizations (partially realize the "$\mathbb{F}_1$-philosphy")

(commutative) monoids, monads, semirings, hyperrings, $\mathbb{S}$-algebras, blueprints, Λ-rings etc.

most basic with "$\mathbb{F}_1$":

+	0	1
0	0	1
1	1	1

? allows idempotent semirings
$\mathbb{B}$, $\mathbb{Z}_{max}$, $\mathbb{R}_{max}$, etc.
related to tropical geometry / arithmetic or scaling site

$\overset{?}{\longleftrightarrow}$ archimedian places

(III) Connes - Consani's $\mathbb{S}$-algebras

Segal's Γ-object $\mathcal{C}$: a "category" where "equivalences" make sense.

<u>Def</u> • Γ-object in $\mathcal{C}$: A $\underbrace{\text{functor}}_{\text{pointed}}$ $X : \mathrm{Fin}_* \longrightarrow \mathcal{C}$ $\quad (\Gamma = \mathrm{Fin}_*^{op})$

$$\{*, 1, \cdots, n\} \mapsto X_n$$

• X is special if the map $X_n \longrightarrow (X_1)^{\times n}$: equiv for $n \geq 0$

• X is very special if special &

the shear map $X_2 \longrightarrow X_1 \times X_1 \simeq X_2$ is an equivalence

"$(x, y) \mapsto (x, xy)$"

<u>Prop</u> $\{\text{special } \Gamma\text{-objects}\} \simeq \mathrm{CMon}(\mathcal{C})$

$\{\text{very special } \Gamma\text{-objects}\} \simeq \mathrm{CMon}^{gp}(\mathcal{C})$

grouplike commutative monoids in $\mathcal{C}$

Corresponding to the objects of interest, one can define various model structures on Γ-sSet (or Γ-Set)

	(0,0)-cat	(∞,0)-cat	(∞,1)-cat	??
$\Gamma \backslash \mathcal{C}$	$\mathrm{Set}_*^{\mathrm{triv}}$ (isom)	$\mathrm{sSet}_*^{\mathrm{Quillen}}$ (weak ho equiv)	$\mathrm{sSet}_*^{\mathrm{Joyal}}$ (categorical equiv)	$\mathrm{sSet}_*^{\mathrm{triv}}$ (isom)
very special group	abelian groups $\mathbb{Z}$	grouplike $\mathbb{E}_\infty$-space $\mathbb{S}$ = connective spectrum	"grouplike" SM $(\infty,1)$-cat	simplicial abelian groups
special monoid	abelian monoids $\mathbb{N}$	$\mathbb{E}_\infty$-space $\mathrm{Fin}^{\simeq}$??	Symmetric monoidal $(\infty,1)$-category $\mathrm{Fin}^{\simeq}$??	simplicial abelian monoids
trivial ??	any Γ-set$_*$ $\mathbb{F}\langle 1\rangle$			any $X \in \Gamma$-sSet$_*$ $\mathbb{S} := \mathbb{F}\langle 1\rangle$ $\boxed{\text{Connes–Consani's } \mathbb{S}\text{-mod}}$

localize

group compl.

localize

decategorify

decategorify

$(\mathcal{C}, \wedge)$ sym. mon. $\rightsquigarrow$ sym mon str. on $\Gamma\mathcal{C}$ by the Day convolution

$$(X \otimes Y)\langle n\rangle = \operatorname*{colim}_{\langle k\rangle \wedge \langle \ell\rangle \to \langle n\rangle} X\langle k\rangle \wedge Y\langle \ell\rangle$$

(L)

$\otimes$-unit is (a fibrant repl. of) $\mathrm{Fin}_* \xhookrightarrow[\mathbb{F}\langle 1\rangle]{} \mathrm{sSet}_*$

Connes - Consani defined $\mathbb{S}$-modules as purely combinatorial, point-set objects

$\mathbb{S}$-algebra = algebra object in $\mathbb{S}$-modules (wrt $\otimes$)

* this generalizes many proposed models:

- (pointed) monoid $M \rightsquigarrow$ monoid ring $\mathbb{S}[M] : \langle n \rangle \mapsto M \wedge \langle n \rangle$
- Semiring $R \rightsquigarrow$ Eilenberg-MacLane construction
  $HR\langle n \rangle := \mathrm{Hom}_*(\langle n \rangle, R)$
- non-special Γ-sets can have multi-valued sums: (or empty)

$$\begin{tikzcd} & X\langle 2 \rangle \arrow[dl, "\delta"'] \arrow[dr, "\mu"] & \\ X\langle 1 \rangle \times X\langle 1 \rangle & & X\langle 1 \rangle \end{tikzcd} \qquad x \oplus y = \mu \circ \delta^{-1}(x, y)$$

$\rightsquigarrow$ hyperrings of the form $\underset{\text{ring}}{R} / \underset{\text{multicative group action}}{G}$

C-C managed to define a structure sheaf of $\mathbb{S}$-algebras on $\overline{\operatorname{Spec}\mathbb{Z}}$

<u>Q.</u> Is isom the correct notion of equivalence between absolute algebras?

- In some sense they are "too rigid"

  cf. <u>Thm by Lawson</u>:

  R : commutative ring object in Γ-sSet,

  $\Rightarrow$ Dyer-Lashof operations of positive degree act on $H_0(R;\mathbb{F}_p)$ by zero

  $\Rightarrow$ $\operatorname{Free}_{\mathbb{E}_\infty}(S^0) \in \operatorname{CAlg}(Sp)$ is not modeled by such R

  $$\curvearrowleft \coprod_{n\geq 0} B\Sigma_n = \operatorname{Fin}^{\simeq}$$

- But we don't want to localize too much:
  - Spectral AG = AG$/\mathbb{S}$, already well-developed, does not seem to capture "Arakelov stuff",
  - Semirings only live in the bottom two rows
  - Connes-Consani approach suggests that $\mathbb{S}$ (or $\mathbb{F}_1$?) should be at least as deep as $\mathrm{Fin}^{\simeq}$ in $\mathbb{E}_\infty$-Spaces
    initial $\mathbb{E}_\infty$-semiring space

$$\begin{tikzcd}[arrows=dash]
\mathbb{R} & \mathbb{C} & \mathbb{Q}_p & & \mathbb{F}_p & \\
 & \mathbb{Q} \arrow[ul] \arrow[u] \arrow[ur] \arrow[dr] & & \mathbb{Z}_p \arrow[ul] \arrow[ur] \arrow[dl] & & \mathbb{Z}_{max} \arrow[d] \\
ku \arrow[dr] & & \mathbb{Z} \arrow[dl] \arrow[drr] & & & \mathbb{B} \arrow[dl] \\
 & \mathbb{S} \arrow[dr] & & & \mathbb{N} \arrow[dll] & \\
 & & \mathrm{Fin}^{\simeq} & & \mathbb{F}_1\ ?? &
\end{tikzcd}$$

Anyway, we'll go deeper than $\mathbb{S}$

$\rightsquigarrow$ expect: absolute alg are "natively derived"

(iii) Observation on descent (that also suggests to go homotopical)

going back: "Spec $\mathbb{Z} \to$ Spec $\mathbb{F}_1$ is a curve"

$$\Rightarrow \quad \mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z} \neq \mathbb{Z}$$

$$\Leftrightarrow \quad \mathrm{Mod}_{\mathbb{Z}} \xrightarrow[\text{forget}]{} \mathrm{Mod}_{\mathbb{F}_1} \text{ is not fully faithful.}$$

<u>Slogan</u>: $\mathbb{Z}$-module structure is an extra structure, not a property of $\mathbb{F}_1$-modules

<u>Q.</u> What is that structure?

<u>cf</u> $\quad \mathrm{Mod}_{\mathbb{F}_q[x]} \longrightarrow \mathrm{Mod}_{\mathbb{F}_q}$

$$\underset{\substack{\uparrow \\ \mathbb{F}_q\text{-vect sp}}}{(V} , \underset{\substack{\in \\ \mathrm{End}(V)}}{f)} \longmapsto V$$

Some monoid-based authors imagine "$\mathbb{Z} \simeq \mathbb{F}_1[2, 3, 5, \cdots]$"

$$
\rightsquigarrow \quad \begin{array}{ccc} \mathrm{Mod}_{\mathbb{Z}} & \longrightarrow & \mathrm{Mod}_{\mathbb{F}_1} \\ \Psi & & \Psi \\ \text{???} & & M \end{array}
\qquad \begin{array}{c} \mathbb{N} \\ ? \end{array}
$$

$\uparrow$ M equipped with $\varphi_p \circlearrowright M$ (and some other stuff?)

(Probably this is the idea behind Borger's geometry of Λ-rings)

cf $\mathrm{Mod}_{\mathbb{Z}} \longrightarrow \mathrm{Mod}_{\mathbb{S}}$

Ψ

If a spectrum X is an $H\mathbb{Z}$-module, lots of "power operations" acts on homology groups of X.

$\mathbb{E}_2$-Hopf-Galois descent data with Galois object $\Sigma^\infty \Omega^2 (S^3\langle 3 \rangle)_+$

(Beardsley-Morava)

About my research project

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Z} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{S} & & \mathbb{N} \\
 & \mathrm{Fin}^{\simeq} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

spectral AG ($\mathbb{S}$)

classical AG ($\mathbb{Z}$)

Semiring AG ($\mathbb{N}$)

$\mathrm{Fin}^{\simeq}$ $\longrightarrow$ AG of $\Big\{$ $\mathbb{E}_\infty$-rig spaces $\Big\}$, symmetric monoidal (∞,∞)-categories

$\underbrace{\mathbb{E}_\infty\text{-rig}} = \mathrm{CAlg}(\mathrm{CMon}^{\otimes}(\infty\mathrm{Gpd}))$

<u>Stabilization</u>: In spectral AG (or already in classical $D(\mathbb{Z})$):

$$\mathbb{E}_\infty\text{-groups} = \text{connective spectra}$$
$$\Big\downarrow \text{stabilization (i.e. invert } \Sigma + \Omega)$$
$$\text{spectra}$$

to shift modules without truncation
(complexes)

$X[1][-1] = X^{sp}$

<u>Obstacle</u> · For $\mathbb{E}_\infty$-monoids, $\Omega \circ \Sigma$ is the group completion functor

· $\Sigma : \mathcal{C} \to \mathcal{C}$ being fully faithful already forces $\mathcal{C}$: additive

<u>Slogan</u>: Only groups are deloopable in $(\infty,1)$-categories

("(n,k)-category" = Category with $0, 1, \cdots, n$ morphisms all morphisms of dim $> k$ is invertible)

"Space" = ∞-groupoid
$= (\infty,0)$-category

<u>Baez-Dolan delooping hypothesis</u>

· $\mathbb{E}_1$-monoid space $=$ $(\infty,1)$ Category with $\exists!$ object

$$M \underset{\vec{\Omega}}{\overset{B}{\rightleftarrows}} BM = \underset{*}{\circlearrowleft} M$$

$$\vec{\Omega}\mathcal{C} := \mathrm{End}_{\mathcal{C}}(*)$$

· $\mathbb{E}_2$-monoid space = monoidal $(\infty,1)$-cat with $\exists!$ obj $= (\infty,2)$-cat with $\exists!$ $0, 1$-mor

• monoidal $\overset{(\infty,1)\text{-}}{\text{category}}$ $(\mathbb{E}_1\text{-})$ $=$ $(\infty,2)$-category with $\exists!$ object $\quad \xleftarrow{\vec{\Omega}}$

• braided monoidal $\overset{(\infty,1)\text{-}}{\text{category}}$ $(\mathbb{E}_2\text{-})$ $=$ monoidal $(\infty,2)$-cat with $\exists!$ object $\quad \xleftarrow{\vec{\Omega}}$

$=$ $(\infty,3)$-cat with $\exists!$ obj. & 1-mor $\quad \xleftarrow{\vec{\Omega}}$

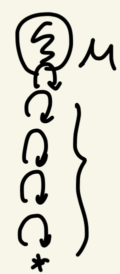
$\Big\}$ generalize

$$\boxed{\begin{array}{l} \mathbb{E}_k\text{-monoidal } (\infty,n)\text{-cat} \leftrightsquigarrow \mathbb{E}_{k-1}\text{-monoidal } (\infty,n+1)\text{-cat with } \exists!\text{ obj} \\ \qquad\qquad\qquad\qquad \leftrightsquigarrow \mathbb{E}_{k-2}\text{-monoidal } (\infty,n+2)\text{-cat with } \exists!\ 0\ \&\ 1\text{-mor} \\ \qquad\qquad\qquad\qquad \leftrightsquigarrow \quad ---- \\ \qquad\qquad\qquad\qquad \leftrightsquigarrow (\infty,n+k)\text{-cat with } \exists!\ 0,1,\dots,(k-1)\text{-mor.} \end{array}}$$

Symmetric monoidal (∞,n)-cat $\quad \mathcal{M}$
$(=\mathbb{E}_\infty)$

$\updownarrow \qquad\qquad \Big\uparrow \vec{\Omega}^k$

Symmetric monoidal $(\infty,n+k)$-cat $\quad B^k\mathcal{M}$



$\exists!\ 0,1,\dots,(k-1)$-mor

In (∞,∞)-Cat, Commutative monoids are infinitely deloopable

: Categorified version of connective spectra

$$\text{“}\ \mathrm{CMon}((\infty,\infty)\mathrm{Cat}) = \infty\mathrm{Sp}^{\mathrm{cn}}\ \text{”}$$

<u>Stabilization</u>

$$\infty\mathrm{Sp} = \lim\left(\cdots \xrightarrow[\vec{\Omega}]{} (\infty,\infty)\mathrm{Cat}_* \xrightarrow[\vec{\Omega}]{} (\infty,\infty)\mathrm{Cat}_*\right)$$

$$\cup$$

$$\infty\mathrm{Sp}^{\mathrm{cn}} = \mathrm{CMon}((\infty,\infty)\mathrm{Cat})$$

$$\left(\begin{array}{l} \text{cf. } \mathrm{Sp} = \lim\left(\cdots \longrightarrow (\infty,0)\mathrm{Cat}_* \xrightarrow[\Omega]{} (\infty,0)\mathrm{Cat}_*\right) \\ \qquad\ \cup \\ \quad \mathrm{Sp}^{\mathrm{cn}} = \mathrm{CMon}^{\mathrm{gp}}((\infty,0)\mathrm{Cat}) \end{array}\right)$$

$$\rightsquigarrow \{\text{(c) rig } (\infty,\infty)\text{-category}\} = \text{(c)}\mathrm{Alg}(\infty\mathrm{Sp})$$

$$\| \qquad\qquad$$

$$\infty\mathrm{Rig}\ /\ \infty\mathrm{CRig}$$

: my proposal for (derived) absolute algebras

This process is implicitly in Connes - Consani's papers:

For $X \in sSet_*$, define $\Omega(X,*) \in sSet_*$ by $\mathrm{Hom}^R_X(*,*)$

where $\left(\mathrm{Hom}^R_X(x,y)\right)_n = \left\{ \begin{tikzcd} \Delta^{n+1} \arrow[r] & X \\ \Delta^n \sqcup \Delta^0 \arrow[u, hook] \arrow[ur, "{(\mathrm{const}_x,\, y)}"'] & \end{tikzcd} \right\}$

$$\begin{cases} X : (\infty,1)\text{-cat} \Rightarrow \mathrm{Hom}^R_X : (\infty,0)\text{-cat} \\ X : (\infty,2)\text{-cat} \Rightarrow \mathrm{Hom}^R_X : (\infty,1)\text{-cat} \end{cases}$$

$$\rightsquigarrow \underset{\mathrm{End}_X(*)}{\underset{\parallel}{\Omega(-,*)}} \text{ models } (\infty,2)\mathrm{Cat}_* \xrightarrow[\vec{\Omega}]{} (\infty,1)\mathrm{Cat}_* \xrightarrow[\vec{\Omega}]{} \overset{\mathrm{Space}_*}{\overset{\parallel}{(\infty,0)}}\mathrm{Cat}_*$$

Unknown if $X \in sSet$ models (∞,n)-cat for $n \geq 3$, so the meaning of $(\Omega(-,*))^{\geq 3}$ is less clear.

(iii) Other justifications

- This is a categorified version of SAG, which is interesting on its own.

  Balmer spectrum

  relation to tensor-triangulated geometry = geometry of stable monoidal $(\infty,1)$-cats "2-rings"
  will be interesting to think about.

- Delooping of comm. monoids or SM $(\infty,1)$-cat are not the only examples of SM(∞,∞)-Cat (or ∞-Spectrum) in nature

  e.g. cobordism category of various flavors

- Group completion of spaces is a subtle operation.

  $\rightsquigarrow$ avoiding it when treating E_n-monoids, etc.

  potentially make things simpler (even for classical loop space theory)

<u>Thm</u> $\vec{\Omega}^n \vec{\Sigma}^n = \mathrm{Free}_{E_n}$ on $(\infty,\infty)\mathrm{Cat}_*$

③ Current attempts to build the geometry of $\mathbb{E}_\infty$-rig spaces / rig (∞,∞)-categories

Want to define scheme-like objects $(\mathcal{X}, \mathcal{O})$

$\left.\begin{array}{ll} \mathcal{X} & \text{a } (\infty\text{-})\text{topos} \\ \mathcal{O} & \text{structure sheaf of algebras} \end{array}\right\}$ s.t. locally equivalent to a "Spectrum" of an algebra

This is based on a site of algebras: classical AG : $\mathrm{CAlg}(\mathrm{Set})^{op, Zar}$
spectral AG : $\mathrm{CAlg}(\mathrm{Sp})^{op, \acute{E}t}$

requires to understand $\left\{\begin{array}{l} \cdot \text{ flatness condition} \\ \cdot \text{ deformation theory} \end{array}\right.$

③ Understanding flatness: Lazard's theorem

<u>Thm</u> (Lazard) $R \in \mathrm{Alg}(\mathrm{Ab})$, $M \in \mathrm{RMod}_R(\mathrm{Ab})$. TFAE:

(1) $M \underset{R}{\otimes} - : \mathrm{LMod}_R(\mathrm{Ab}) \longrightarrow \mathrm{Ab}$ is left exact.

(2) M is a filtered colimit of free modules of finite rank.

<u>Thm</u> (Katsov) Ab replaced by $\mathrm{CMon}(\mathrm{Set})$ (so R: semiring)

<u>Thm</u> (Lurie) Ab replaced by Sp^{cn} (R: connective ring spectrum)

<u>Thm</u> (M.) · Ab replaced by $\mathrm{CMon}(\mathrm{Space})$ (R: rig space)

· all of these follows from the following categorical result:

<u>Setting</u>

$\mathcal{C}$: small $\overset{(1,1)}{(\infty,1)}$-category with finite coproducts

product-preserving

$P(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Space})$ $\quad P_\Sigma(\mathcal{C}) = \mathrm{Fun}^\times(\mathcal{C}^{op}, \mathrm{Space})$

$F : \mathcal{C} \longrightarrow \overset{\mathrm{Set}}{\mathrm{Space}}$

Let
$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, hook, "j"] \arrow[dd, bend right, hook, "\sharp"'] & \mathrm{Space} \\
\mathcal{P}_\Sigma(\mathcal{C}) \arrow[ur, "\mathrm{lan}_j F"'] \arrow[d, hook] & \\
\mathcal{P}(\mathcal{C}) \arrow[uur, "\mathrm{lan}_\sharp F"'] &
\end{tikzcd}$$
be the left Kan extensions

then TFAE: (1) Let
$$\begin{tikzcd}
\int F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Space}_* \arrow[d] \\
\mathcal{C} \arrow[r] & \mathrm{Space}
\end{tikzcd}$$
, then $\int F$ is cofiltered

(2) F is a filtered colimit of corepresentable functors

(3) $\mathrm{Lan}_\sharp F : \mathcal{P}(\mathcal{C}) \longrightarrow \mathrm{Space}$ is left exact

(4) $\mathrm{Lan}_j F : \mathcal{P}_\Sigma(\mathcal{C}) \longrightarrow \mathrm{Space}$ is left exact

(Our Case: $\mathcal{C} = \{\text{finite free left } R\text{-modules}\} \rightsquigarrow \mathcal{P}_\Sigma(\mathcal{C}) = \mathrm{LMod}_R$

$F : \mathcal{C} \longrightarrow \mathrm{CMon} \xrightarrow[\text{forget}]{} \mathrm{Space} \quad \longleftrightarrow \quad M \in \mathrm{RMod}_R$

$R^n \longmapsto M^n$)

$\rightsquigarrow$ It seems safe to define flatness as these equivalent conditions.

- This can be used to define e.g.

  $A \xrightarrow{f} B$ in CRig(Spaces) is <u>weakly étale</u>

  if f is flat and $\Delta_f : B \to B \otimes_A B$ is flat

  - used in pro-étale site paper by Bhatt-Scholze
  - for ring spectra, with a mild finiteness condition, this implies étale.

- I should prove fpqc descent of modules

③ Deformation theory

(Notation: $L_A := L_{A/\mathbb{S}}$)

- Cotangent complex $L_{B/A}$ is crucial in SAG.

  (for $A \to B$ in $\mathrm{CAlg}(\mathrm{Sp})$)

  - Used to describe the obstruction to deform maps / algebras along nilpotent thickenings
  - $\mathrm{Spec}\,\pi_0 A \longrightarrow \mathrm{Spec}\,A$ is a nilpotent thickening

- From "absolute mathematics" PoV, we're interested in "differentiating numbers"

  - Fermat quotient $\partial_p = \dfrac{(-)^p - (-)}{p}$
  - "absolute derivation" of Kurokawa-Ochiai-Wakayama $\begin{cases} \dfrac{\partial}{\partial p} p^n = n \cdot p^{n-1} \\ \dfrac{\partial}{\partial p} \ell = 0 \end{cases}$

    ↑ only Leibniz rule, no additivity

"$L_{\mathbb{Z}/\mathbb{F}_1}$" is where differentials of $n \in \mathbb{Z}$ is supposed to live.

Construction of cotangent complexes in SAG:

A : ring spectrum

<u>key fact</u> $\quad \mathrm{Mod}_A \simeq \mathrm{Sp}(\mathrm{CAlg}_{/A}) \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathrm{CAlg}_{/A}$

$\uparrow$ stabilization, i.e. invert $\Sigma \dashv \Omega$

- $M \longmapsto \begin{array}{c} A \oplus M \\ \downarrow \\ A \end{array}$ split square-zero ext

- $\operatorname*{colim}_{n\to\infty} \left[\Omega^n I \; \Sigma^n \begin{pmatrix} B \otimes A \\ \uparrow\downarrow \\ A \end{pmatrix} \right] \longleftarrow\!\!\mid \begin{array}{c} B \\ \downarrow \\ A \end{array}$

  I: aug ideal; Σ^n: in $\mathrm{CAlg}^{\mathrm{aug}}_{/A}$

Define $L_A := \Omega^\infty(\mathrm{id}_A)$

<u>Use</u> the stability of Mod_A + some elementary Goodwillie calculus

Imitating this in (∞,∞)-Setting :

<u>conj</u> $\mathrm{Mod}_A(\infty\mathrm{Sp}) \simeq \infty\mathrm{Sp}(\infty\mathrm{Rig}_{A/}) := \lim\left(\cdots \xrightarrow[\vec{\Omega}]{} \infty\mathrm{Rig}_{A/} \xrightarrow[\vec{\Omega}]{} \infty\mathrm{Rig}_{A/}\right)$

- "stability" of $\mathrm{Mod}_A(\infty\mathrm{Sp})$ ✓

  $\Leftarrow$ $\vec{\Omega} \dashv \vec{\Sigma}$ is invertible

- the rest seems likely to work, but not yet worked out.

If this is true. <u>Def</u> $\infty\mathrm{Rig}_{A/} \longrightarrow \mathrm{Mod}_A(\infty\mathrm{Sp})$

$$\mathrm{id}_A \longmapsto L_A$$

<u>Q</u>. What kind of "nilpotent extensions" does it classify ?

- Obstruction to the deformation $\rightsquigarrow$ relation to "Witt construction in char 1" ?
- Compute $L_{\mathbb{Z}}, L_{\mathbb{N}}$, etc. what kind of "differentials" live there ?