NT learning Seminar Dec 02 / 2020

Deligne - Illusie "Relèvements modulo p^2 et décomposition du complexe de de Rham"

- (i) Introduction, Hodge-to-de Rham degeneration, char $p \rightsquigarrow$ char 0
- (ii) Frobenius, statement of the main thm
- (iii) Cartier isomorphism
- (iv) Proof of the main thm
- ((v) Better formulation by gerbes)

(1) Introduction : Hodge-to-de Rham degeneration

X : smooth proper scheme / a field k

$\rightsquigarrow \Omega^\bullet_{X/k}$ (algebraic) de Rham complex

$\rightsquigarrow H^n_{dR}(X/k) := H^n(X; \Omega^\bullet_{X/k})$

if $k = \mathbb{C}$ — holomorphic

$\underset{\text{GAGA}}{\simeq} H^n(X^{an}; \Omega^\bullet_{X^{an}}) \simeq H^n(X^{an}; \mathbb{C})$

Poincaré lemma $\mathbb{C} \xrightarrow[\text{qis}]{\sim} \Omega^\bullet_X$

"cyclotomic spec"

$\mathbb{T} \curvearrowright THH^{t\mathbb{T}} = TP$

$HH \qquad P$

$\mathcal{C} = \mathrm{Perf}(X)$

Filtration on $\Omega^\bullet_{X/k}$ by stupid truncation

$$(\mathrm{Fil}^i \Omega_{X/k})^n := \begin{cases} \Omega^n_{X/k} & \text{if } n \leq i \\ 0 & \text{otherwise} \end{cases}$$

$\rightsquigarrow$ Hodge-to-de Rham SS

$$E_1^{i,j} = H^j(X, \Omega^i_{X/k}) \Longrightarrow H^{i+j}_{dR}(X/k)$$

In particular : $E_\infty^{i,j}$: subquotient of $E_1^{i,j}$,

$\bigoplus_{i+j=n} E_\infty^{i,j}$ is the assoc. gr. of a filtration on $H^n_{dR}(X/k)$

When $k = \mathbb{C}$, Hodge decomposition : $H^n_{dR}(X/k) \simeq \bigoplus_{i+j=n} H^j(X, \Omega^i_{X/k})$
(& X projective) (use analysis)

i.e. $E_1^{i,j} = E_\infty^{i,j}$ & the filtration on $H^n_{dR}(X/k)$ splits.

↑ we say SS **degenerates** at E_1 page (purely algebraic question)

Set $h^{i,j} := \dim_k E_1^{i,j}$, $h^n := \dim_k H^n_{dR}(X/k)$ ($< \infty$ by $\Omega^i_{X/k}$: coherent)

$$\rightsquigarrow h^n = \sum_{i+j=n} \dim_k E_\infty^{i,j} \le \sum_{i+j=n} \dim_k E_1^{i,j} = \sum_{i+j=n} h^{i,j}$$

so SS degen at E_1 $\iff$ $h^n = \sum_{i+j=n} h^{i,j}$

__Cor 2.7__ K : field of char 0, X : sm. proper $/K$ $\Rightarrow$ SS degen. at E_1

__Cor 2.4__ k : perfect of char $p > 0$, X/k : sm. proper, $\dim \le p$.

If $\exists \tilde{X}$ s.t.
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{X} \arrow[d, "\text{smooth}"] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} W_2(k) \end{tikzcd}$$
, then SS degen at E_1.

Cor 2.4 $\Rightarrow$ Cor 2.7 "From char $p>0$ to char 0" (sketch)

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{X} \arrow[d, "f"', "\text{sm proper}"] & X \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\
\operatorname{Spec} k \arrow[r, hook] & S = \operatorname{Spec} A \arrow[d, "\text{sm.}"] & \operatorname{Spec} K \arrow[l] \\
& \operatorname{Spec} \mathbb{Z} &
\end{tikzcd}$$

- $K = \varinjlim_{\substack{A \subset K \\ \text{fin.gen } \mathbb{Z}\text{-alg}}} A$
- $X \xrightarrow{f_0} \operatorname{Spec} K$ is of fin pres.

$\rightsquigarrow$ it is a base change of $\exists\, \mathfrak{X} \xrightarrow{f} \operatorname{Spec} A =: S$

X: sm. proper $\Rightarrow$ $\mathfrak{X} \longrightarrow \operatorname{Spec} A$ can be taken smooth proper

- Restricting to an open set $\operatorname{Spec} A[s^{-1}]$ if necessary, we may assume
  - $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ smooth (A integral fin type + "generic smoothness")
  - $R^j f_* \Omega^i_{\mathfrak{X}/S}$ and $R^n f_* \Omega^\bullet_{\mathfrak{X}/S}$ are locally free (stalk @ $\operatorname{Spec} K$: free of fin type)
- Take a bound d of dim of fibers of $\mathfrak{X}$, $N = \prod (\text{prime} \leq d)$
  - S: Jacobson ($\Leftarrow$ fin type $/\mathbb{Z}$) $\rightsquigarrow$ $\operatorname{Spec} A[1/N] \subset S$ contains a closed pt $\operatorname{Spec} k \longrightarrow S$
  - $k \simeq \mathbb{F}_{p^m}$, $p > d$

- $$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d, "\text{nilp. thickening}"'] & S \arrow[d, "\text{smooth}"] \\ \operatorname{Spec} W_2(k) \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z} \end{tikzcd} \rightsquigarrow \begin{tikzcd} Y \arrow[r] \arrow[d, "\substack{\text{sm. proper} \\ \dim < p}"'] \arrow[dr, phantom, "\lrcorner"] & \widetilde{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{X} \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} W_2(k) \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

Apply Cor 2.4 $\rightsquigarrow \displaystyle\sum_{i+j=n} h^{i,j}(Y/k) = h^n(Y/k)$

- By (a generalization of) flat base change

$$\left(\begin{tikzcd} X_0 \arrow[r, "g"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ S_0 \arrow[r, "g"'] & S \end{tikzcd} \right. \quad Lg^* Rf_* \mathcal{F} \xrightarrow{\sim} Rf_{0*} g^* \mathcal{F} \quad \text{q.i.s}$$

e.g. if S : noetherian integral,
f : smooth proper,
$\mathcal{F}$: locally free $\mathcal{O}_X$-mod of finite type

$\xrightarrow[\text{argument}]{+\text{ some}}$ $h^{i,j}(Y/k) = \mathrm{rk}_A(R^j f_* \Omega^i_{\mathfrak{X}/S}) = h^{i,j}(X/K)$

$h^n(Y/k) = \mathrm{rk}_A(R^n f_* \Omega^\bullet_{\mathfrak{X}/S}) = h^n(X/K)$

$\rightsquigarrow \displaystyle\sum_{i+j=n} h^{i,j}(X/K) = h^n(X/K)$.

$\boxtimes$

Frobenius

Absolute Frobenius : X scheme$/\mathbb{F}_p$ $\rightsquigarrow$ $F_X : (|X|, \mathcal{O}_X) \longrightarrow (|X|, \mathcal{O}_X)$

$$\begin{cases} \cdot\ \mathrm{id} \text{ on } |X|, \\ \cdot\ \mathcal{O}_X \xrightarrow[(-)^p]{} \mathrm{id}_* \mathcal{O}_X = \mathcal{O}_X \end{cases}$$

Relative Frob $X \xrightarrow{f} S$ $/\mathbb{F}_p$

$$\begin{tikzcd}
X \arrow[rr, bend left, "F_X"] \arrow[r, dashed, "F"] \arrow[dr, "f"'] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \circlearrowleft"] & X \arrow[d, "f"] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
X \arrow[rr, "F = F_{X/S}"] \arrow[dr] & & X' \arrow[dl] \\
& S &
\end{tikzcd}$$

- F_S : universally homeo
$\Longrightarrow F_{X/S}$ homeo

examples ① when $X = \mathrm{Spec}\, \mathcal{O}_S[t_1, \dots, t_d] = \mathbb{A}^d_S$

$$\begin{tikzcd}
\mathcal{O}_S[t_1, \dots, t_d] & & \mathcal{O}_S[t_1, \dots, t_d] \arrow[ll, "(-)^p"'] \arrow[dl, "\text{coef } (-)^p"] \\
& \mathcal{O}_S[t_1^p, \dots, t_d^p] \arrow[ul] & \\
\mathcal{O}_S \arrow[uu] \arrow[ur] & & \mathcal{O}_S \arrow[ll, "(-)^p"] \arrow[uu]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_S[t_1, \dots, t_d] & & \mathcal{O}_S[\tilde{t}_1, \dots, \tilde{t}_d] \arrow[ll] \\
t_i^p & & \tilde{t}_i \arrow[ll, mapsto] \\
& \mathcal{O}_S \arrow[uul] \arrow[uur] &
\end{tikzcd}$$

② when f: étale : F is also étale. $\xrightarrow[\text{+ homeo}]{F \text{ induce radical residue field ext}}$ F : isom

③ When f: smooth : locally
of rel dim d

$$\begin{tikzcd} X \arrow[r, "\text{étale}"] \arrow[dr, "f"'] & \mathbb{A}^d_Y \arrow[d, "pr"] \\ & Y \end{tikzcd}$$

Combine ① + ②

$\rightsquigarrow \begin{cases} F : X \to X' \text{ finite flat}, \\ F_* \mathcal{O}_X : \text{locally free of rank } p^d \ / \ \mathcal{O}_{X'} \end{cases}$

③ + $\Omega^i_{X/S}$: locally free $/ \mathcal{O}_X$ of rank $\binom{d}{i}$

$\rightsquigarrow F_* \Omega^i_{X/S}$: locally free of $p^d \cdot \binom{d}{i}$ $/ \mathcal{O}_{X'}$

④ Lift to mod p^2

$$\begin{tikzcd} X \arrow[d, "f"'] & \\ S \arrow[r, hook, "i"'] & \tilde{S} \end{tikzcd}$$

- f : smooth (resp. flat)
- i : thickening of order 1

$0 \to I \to \mathcal{O}_{\tilde{S}} \to i_* \mathcal{O}_S \to 0, \quad I^2 = 0$

Def A smooth (flat) lift of f to $\tilde{S}$ is a smooth (flat) $\tilde{S}$-scheme $\tilde{X}$ with a pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & \tilde{X} \arrow[d, "\tilde{f}"] \\ S \arrow[r] & \tilde{S} \end{tikzcd}$$

Fact

- $\exists$ obstruction class $\omega(f) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_{X/S}, f^*I)$

  $\exists\ \tilde{X} \xrightarrow{\tilde{f}} \tilde{S} \iff \omega(f) = 0$

- $\{\tilde{f} : \tilde{X} \to \tilde{S}\} /_{\tilde{S}\text{-isom}}$: $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, f^*I)$ – torsor

- $\mathrm{Aut}(\tilde{X} \xrightarrow{\tilde{f}} \tilde{S}) = \mathrm{Hom}(\Omega^1_{X/S}, f^*I)$

(More concisely: $\{$space of $\tilde{X} \xrightarrow{\tilde{f}} \tilde{S}\} = \mathrm{Map}_{\mathcal{O}_X}(\cancel{\Omega^1_{X/S}}\ L_{X/S}, f^*I[2])$)

RHom

If f : smooth, $\Omega^1_{X/S}$: locally free ($\Rightarrow \mathrm{Ext}^\bullet(\Omega^1_{X/S}|_U, -) = 0$)

$\rightsquigarrow$ locally $\tilde{X}$ exists

(iii) Statement of the main theorem

Setting :

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, "f"'] & X' \arrow[d] \arrow[r, dashed] & \tilde{X}' \arrow[d, dashed] \\ & S \arrow[d] \arrow[r, hook] & \tilde{S} \arrow[d, "\text{flat}"] \\ & \operatorname{Spec} \mathbb{F}_p \arrow[r, hook] & \operatorname{Spec} \mathbb{Z}/p^2 \end{tikzcd}$$

smooth $\to f$

special case Thm 2.1: $\operatorname{Spec} k = S \hookrightarrow \tilde{S} = \operatorname{Spec} W_2(k)$ for k : perfect field

Cor 3.6 a lift $\tilde{X}'$ of X' over $\tilde{S}$ / isom

$\overset{1:1}{\longleftrightarrow}$ multiplicative isom $\bigoplus_{i<p} \mathcal{H}^i F_* \Omega^\bullet_{X/S}[-i] \xrightarrow{\sim} \tau_{<p} F_* \Omega^\bullet_{X/S}$ in the derived cat $D(X')$

Cor 2.7 k : perfect of char p, X/k : smooth proper, $\dim < p$

If X admits a lift $\tilde{X}/W_2(k)$, then the SS

$$E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Longrightarrow H^{i+j}_{dR}(X/k) \text{ degenerates at } E_1.$$

Cor 2.1 $\Longrightarrow$ Cor 2.7

$$\bigoplus_j H^j\Big(X', \bigoplus_{i<p} \underbrace{\mathcal{H}^i F_*\Omega^\bullet_{X'/k}}_{\cong\ \Omega^i_{X'/k}}[-i]\Big) \xrightarrow[\text{by } 2.1]{\sim} \bigoplus_n H^n(X', F_*\Omega^\bullet_{X/k})$$

Cartier isom (later)

$\Omega^i_{X'/k}[-i]$

$$\bigoplus_n H^n(X', F_*\Omega^\bullet_{X/k}) \cong \bigoplus H^n(X, \Omega^\bullet_{X/k}) \qquad \longleftarrow F_* : \text{homeo}$$

$$\bigoplus H^n(X, \Omega^\bullet_{X/k}) = \bigoplus H^n_{dR}(X/k)$$

$\dim = h^n$

deg n part

$$\bigoplus_{\substack{i+j=n \\ i<p}} H^j(X', \Omega^i_{X'/k})$$

$\Omega^i_{X'/k} \sim F_S^*\Omega^i_{X/k}$

$$\cong \bigoplus H^j(X, \Omega^i_{X/k})$$

$$\dim = \sum_{\substack{i+j=n \\ i<p}} h^{ij}$$

$$\begin{tikzcd} X' \arrow[r, "F_S"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ S \arrow[r, "\cong"] & S \end{tikzcd} \Longrightarrow F_S : \text{isom}$$

S: perfect

If $\dim X < p$, then $H^{\geq p}(X, (\text{coh sheaf})) = 0$

$$\rightsquigarrow h^n = \sum_{i+j=n} h^{ij}$$

∎

11 Key tool : Cartier isomorphism

<u>Setting</u>

$$\begin{tikzcd}
X \arrow[dr, "F"] \arrow[drr, bend left, "Fr"] \arrow[ddr, bend right] & & \\
& X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
& S \arrow[r, "Fr"'] & S
\end{tikzcd}$$

as before

<u>Note</u>

$$\begin{tikzcd}
\Omega^1_{X'/S} \arrow[r, "F^\# = 0"] & F_*\Omega^1_{X/S} & 0 = \cancel{p} \cdot s^{p-1} ds \\
s \in \mathcal{O}_{X'} \arrow[u, "d"] \arrow[r, "F^\#"'] & F_*\mathcal{O}_X \arrow[u, "d"'] & s^p \arrow[u, mapsto]
\end{tikzcd}$$

$s \longmapsto s^p$

<u>Want</u> " $\frac{1}{p} F^\#$ "

<u>Thm</u> (a) $\exists ! \ \gamma = \bigoplus \gamma^i : \bigoplus_i \Omega^i_{X'/S} \longrightarrow \bigoplus_i H^i F_* \Omega^\bullet_{X/S}$ hom in $grAlg_{\mathcal{O}_{X'}}$

s.t.

$$\begin{tikzcd}
\gamma^0 : \mathcal{O}_{X'} \arrow[r, dashed] \arrow[dr, "F^\#"'] & H^0 F_* \Omega^\bullet_{X/S} \arrow[d, hook] \\
& F_* \mathcal{O}_X
\end{tikzcd}$$

$$\begin{tikzcd}
\gamma^1 : \Omega^1_{X'/S} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^1 F_* \Omega^\bullet_{X/S} \\
\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} & {[s^{p-1} ds]} \arrow[u, phantom, "\in" rotate=90] \\
\Omega^1_{X/S} \ni ds \arrow[u, "1 \otimes (-)"] \arrow[ur, mapsto] & \\
\mathcal{O}_X \ni s \arrow[u, "d"] &
\end{tikzcd}$$

only additive after $[-]$

(b) γ : isom if f : smooth
(denoted by C^{-1})

(important)
<u>example</u>

$$\begin{array}{ccc} X & = & \operatorname{Spec} \mathbb{F}_p[t] \\ \downarrow & & \downarrow \\ S & = & \operatorname{Spec} \mathbb{F}_p \end{array}$$

$$\begin{array}{ccccc} t^p & \longleftarrow\!\!\!\shortmid & \tilde{t} & & \\ \cap & & \cap & & \\ \mathbb{F}_p[t] & \xleftarrow{F^\#} & \mathbb{F}_p[\tilde{t}] & \longleftarrow & \mathbb{F}_p[t] \\ & & \| & & \uparrow \\ & & t^p & & (\text{coef})^p \end{array}$$

$\rightsquigarrow \gamma : \mathcal{O}_{X'} \oplus \Omega^1_{X'/S} \longrightarrow \mathcal{H}^0 F_* \Omega^\bullet_{X/S} \oplus \mathcal{H}^1 F_* \Omega^\bullet_{X/S}$

$\mathbb{F}_p[\tilde{t}] \xrightarrow[\cong]{F^\#} \mathbb{F}_p[t^p] \hookrightarrow \mathbb{F}_p[t]$, $\quad \mathcal{H}^0 = \mathbb{F}_p[t^p]$

$$\begin{array}{c} 0 \\ \downarrow \\ \mathbb{F}_p[t] \\ \downarrow \\ \mathbb{F}_p[t](dt) \\ \downarrow \\ 0 \end{array} \qquad \begin{array}{c} f(t) \\ \downarrow \\ \frac{df}{dt}\, dt \end{array}$$

$\mathbb{F}_p[t] \to \langle t^k \cdot dt \mid p \nmid k+1 \rangle \hookrightarrow \mathbb{F}_p[t](dt)$

$t^{mp} \longmapsto t^{p-1}$, $\quad mp-1$, $\quad 0$

$\oplus$

$\mathbb{F}_p[\tilde{t}](d\tilde{t}) \xrightarrow{\cong} \langle t^{pm-1} dt \mid m \in \mathbb{N} \rangle = \mathcal{H}^1$

$d\tilde{t} \longmapsto t^{p-1} \cdot dt$

$\tilde{t} \longmapsto t^p$

(iii) Proof of the main theorem

Cor 3.6 a lift $\tilde{X}'$ of X' over $\tilde{S}$ / isom $\cancel{\longleftrightarrow}$ isom $\bigoplus_{i<p} \mathcal{H}^i F_*\Omega^\bullet_{X/S}[-i] \longrightarrow F_*\Omega^\bullet_{X/S}$ in the derived cat $D(X')$ s.t. (induces id on $\mathcal{H}^i$)

only prove this

$$\begin{tikzcd}
\bigoplus_{i<p} \mathcal{H}^i F_*\Omega^\bullet_{X/S}[-i] \arrow[r] & F_*\Omega^\bullet_{X/S} \\
\bigoplus_{i<p} \Omega^i_{X'/S}[-i] \arrow[u, "C^{-1}"] \arrow[ur, dashed, "\bigoplus \varphi^i"'] &
\end{tikzcd}$$

Setting:

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[dr, "f"', "\text{smooth}"] & X' \arrow[d] \arrow[r, dashed] & \tilde{X}' \arrow[d, dashed] \\
& S \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tilde{S} \arrow[d, "\text{flat}"] \\
& \operatorname{Spec} \mathbb{F}_p \arrow[r, hook] & \operatorname{Spec} \mathbb{Z}/p^2
\end{tikzcd}$$

It suffices to construct φ^i s.t. $\mathcal{H}^i(\varphi^i) = C^{-1}$ (in the derived cat)

Step 1 enough to construct φ^1

$$\begin{tikzcd}
(\Omega^1_{X'/S}[-1])^{\otimes^L i} \arrow[r, "(\varphi^1)^{\otimes^L i}"] \arrow[d, "\cong"] & (F_*\Omega^\bullet_{X/S})^{\otimes^L i} \arrow[d, "\cong"] \\
(\Omega^1_{X'/S})^{\otimes i}[-i] \arrow[d, shift left] \arrow[r, phantom, "\circlearrowright"] & (F_*\Omega^\bullet_{X/S})^{\otimes i} \arrow[d, "\text{product}"] \\
\Omega^i_{X'/S}[-i] \arrow[u, shift left] \arrow[r, "\varphi^i"] & F_*\Omega^\bullet_{X/S}
\end{tikzcd}$$

(locally free $\rightsquigarrow \otimes^L = \otimes$)

$\omega_1 \otimes \cdots \otimes \omega_i \mapsto \omega_1 \wedge \cdots \wedge \omega_i$

Section $\frac{1}{i!}\sum_\sigma \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)} \mapsfrom \omega_1 \wedge \cdots \wedge \omega_i$ $\boxed{i<p}$

$\rightsquigarrow$ • Condition on φ^1 : $[\varphi^1\omega] = C^{-1}\omega$

• $[\varphi^i(\omega_1\wedge\cdots\wedge\omega_i)] = [\varphi^1\omega_1]\wedge\cdots\wedge[\varphi^1\omega_i]$

$C^{-1}(\omega_1\wedge\cdots\wedge\omega_i) = C^{-1}\omega_1\wedge\cdots\wedge C^{-1}\omega_i$

(From now on $\varphi := \varphi^1$)

Step 2

Recall Setting :

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[dr, "F"] \arrow[dd, "f"', "\text{smooth}"] & \tilde X \arrow[dr, "\tilde F"] & \\
& X' \arrow[r, dashed] \arrow[dl] & \tilde X' \arrow[dl, dashed] \\
S \arrow[r, hook] \arrow[d] & \tilde S \arrow[d, "\text{flat}"] & \\
\operatorname{Spec}\mathbb{F}_p \arrow[r, hook] & \operatorname{Spec}\mathbb{Z}/p^2 &
\end{tikzcd}$$

Assume $\exists$ lift of F

$\Downarrow$

construct

$\varphi_{\tilde F} : \Omega^1_{X'/S} \longrightarrow F_*\Omega^1_{X/S}$

• multiplication by p : $\Omega^1_{X/S} \xrightarrow{\sim} p\,\Omega^1_{\tilde X/\tilde S}$ isom of $\mathcal{O}_X$-modules

[over $\mathbb{Z}/p^2$, killed by $p\mathbb{Z}/p^2\mathbb{Z}$

$\rightsquigarrow$ $\mathcal{O}_{\tilde X}$-mod structure factors through $\mathcal{O}_X$-mod str., isom to mod p reduction

$$\begin{tikzcd}
\tilde F^\# : \Omega^1_{\tilde X'/\tilde S} \arrow[r] \arrow[d] & \tilde F_*\Omega^1_{\tilde X/\tilde S} \\
\Omega^1_{X'/S} \arrow[r] & p\tilde F_*\Omega_{\tilde X/\tilde S} \arrow[u, hook] \\
& F_*\Omega^1_{X/S} \arrow[u, "\simeq"]
\end{tikzcd}$$

C^{-1} Z^1 H^1

$$\Omega^1_{\tilde X'/\tilde S} \xrightarrow{\tilde F^\#} \tilde F_* \Omega_{\tilde X/\tilde S}$$

$\mathcal{O}_X$

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "F"] \arrow[ddr, "f"'] & \tilde X \arrow[dr, "\tilde F"] & \\
& X' \arrow[r] \arrow[d] & \tilde X' \arrow[d] \\
& S \arrow[r] \arrow[d] & \tilde S \arrow[d] \\
& \operatorname{Spec} \mathbb{F}_p \arrow[r] & \operatorname{Spec} \mathbb{Z}/p^2
\end{tikzcd}$$

(a) $\cdot p : \tilde F_* \Omega^1_{X/S} \xrightarrow{\simeq} p \tilde F_* \Omega^1_{\tilde X/\tilde S}$ of $\mathcal{O}_{X'}$-module. of $\mathcal{O}_X$-module.

(b) $\tilde F^\# : \Omega^1_{\tilde X'/\tilde S} \longrightarrow \tilde F_* \Omega_{\tilde X/\tilde S}$

mod-p reduction ($\Omega^1_{X'/S} \longrightarrow F_* \Omega^1_{X/S}$ is zero)

$\mathcal{O}_{\tilde X}$ $\mathcal{O}_{X'} \otimes_{\mathcal{O}_{\tilde X'}} (-)$ $\mathcal{O}_{\tilde X'}$ (c) $\mathcal{O}_X$ $\mathcal{O}_{X'}$

$$\begin{tikzcd}
\Omega^1_{\tilde X'/\tilde S} \arrow[r] \arrow[d] \arrow[drr, dashed] & \tilde F_* \Omega_{\tilde X/\tilde S} \arrow[r, hook] & \\
& & p \tilde F_* \Omega^1_{\tilde X/\tilde S} \\
\Omega^1_{X'/S} \arrow[rr, dashed, "\varphi_{\tilde F}"] \arrow[drr] \arrow[ddrr, "C^{-1}"'] & & F_* \Omega^1_{X/S} \arrow[u, "\simeq (a)"'] \\
& & Z^1 F_* \Omega^\bullet_{X/S} \arrow[u, hook] \arrow[d] \\
& & H^1 F_* \Omega^\bullet_{X/S}
\end{tikzcd}$$

$\tilde a^p + p \cdot h \longmapsto p(\tilde a^{p-1} d\tilde a + dh)$

$a \longmapsto [a^{p-1} da]$

$$\begin{tikzcd}
\tilde a' & \mathcal{O}_{\tilde X'} \arrow[r, "\tilde F^\#"] \arrow[d, "\bmod p"] & \mathcal{O}_{\tilde X} \ni \tilde a \arrow[r, "d"] \arrow[d, "\bmod p"] & \Omega^1_{\tilde X/\tilde S} \arrow[d] \\
a' & \mathcal{O}_{X'} \arrow[r, "F^\#"] & \mathcal{O}_X \ni a \arrow[r, "d"] & \Omega^1_{X/S} \\
& \mathcal{O}_X \arrow[u] \arrow[ur, "(-)^p"'] & a \mapsto a^p &
\end{tikzcd}$$

~~Proof~~ (c).

Let
$$\begin{tikzcd} X \arrow[r] \arrow[d] & \tilde X \arrow[r, shift left, "\tilde F_1"] \arrow[r, shift right, "\tilde F_2"'] & \tilde X' \\ X' \arrow[rr] & & . \end{tikzcd}$$
two lifts.

$$\begin{tikzcd} \tilde F_2^\# - \tilde F_1^\# : \mathcal{O}_{\tilde X'} \arrow[r] \arrow[d] & p \cdot \tilde F_* \mathcal{O}_X \\ \Omega^1_{X'/S} \arrow[r, "\cancel{h_{12}}"] & \tilde F_* \mathcal{O}_X \arrow[u, "\int"'] \end{tikzcd}$$

$F - F = 0$ ~~$F_2 - F$~~

$\wr$ mod p

$$\begin{tikzcd} \mathcal{O}_{\tilde X'} \arrow[r, "\tilde F_2^\# - \tilde F_1^\#"] \arrow[d] \arrow[dr] & \tilde F_* \mathcal{O}_{\tilde X} \arrow[r, "\tilde F_* d"] & \tilde F_* \Omega^1_{\tilde X/\tilde S} \\ & p \tilde F_* \mathcal{O}_{\tilde X} \arrow[u, hook] & p \tilde F_* \Omega^1_{\tilde X/S} \arrow[u, hook] \\ \mathcal{O}_{X'} \arrow[d] \arrow[r, dashed] & F_* \mathcal{O}_X \arrow[u, "\simeq"] \arrow[r, "F_* d"] & F_* \Omega^1_{X/S} \arrow[u, "\simeq"] \\ \Omega^1_{X'/S} \arrow[ur, "h_{12}"'] \arrow[uuurr] & & \end{tikzcd}$$

h_{12}

$\|$

$\varphi_{\tilde F_2} - \varphi_{\tilde F_1}$.

$$\begin{tikzcd} \mathcal{O}_{\tilde X'} \arrow[r, "\tilde F_2^\# - \tilde F_1^\#"] & p \tilde F_* \mathcal{O}_{\tilde X} \\ & p \arrow[u] \end{tikzcd}$$

(d).

$$\begin{tikzcd}
U_\alpha \arrow[rr] \arrow[d, "j_\alpha"] \arrow[dr, "F"] & & \tilde U_\alpha \arrow[d, "\tilde F_\alpha"] \\
X \arrow[dr] \arrow[ddr] & U'_\alpha \arrow[r] \arrow[d] & \tilde U'_\alpha \arrow[d] \\
& X' \arrow[r] \arrow[d] & \tilde X' \arrow[d] \\
& S \arrow[r] & \tilde S
\end{tikzcd}$$

$$\varphi_{\tilde F_\alpha} : \Omega^1_{X'/S}|_{U'_\alpha} \longrightarrow F_* \Omega^1_{X/S}|_{U'_\alpha}$$

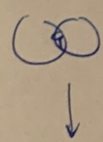
$$\cancel{\check{C}(\mathcal{U}, \Omega^1_{X'/S}) \longrightarrow \check{C}(\mathcal{U}, F_*\Omega^1_{X/S})}$$

$$\cancel{\Omega^1_{X'/S} \longrightarrow F_*\Omega^1_{X/S}}$$

cover $X = \bigcup U_\alpha$

$$U_\alpha \xhookrightarrow{j_\alpha} X$$

$$\begin{tikzcd}
1 \arrow[d] & \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & \coprod_{\alpha\beta} j_{\alpha\beta *} j^*_{\alpha\beta} \Omega_{X'/S} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_\alpha j_{\alpha *} j^*_\alpha \Omega^1_{X'/S} \arrow[r, "\mathrm{colim}"] \arrow[d, "\coprod j_{\alpha *} \varphi_{\tilde F_\alpha}"] & \Omega^1_{X'/S} \arrow[d, dashed] \\
2 & \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & \coprod_{\alpha\beta} j_{\alpha\beta *} j^*_{\alpha\beta} F_*\Omega^1_{X/S} \arrow[r, shift left] \arrow[r, shift right] & \coprod_\alpha j_{\alpha *} j^*_\alpha F_*\Omega^1_{X/S} \arrow[r, "\mathrm{colim}"] & F_*\Omega^1_{X/S}
\end{tikzcd}$$



(b)(c) gives a morphism of "descent data".

$(X, \mathcal{O})$ ringed space

$K^\bullet \in \mathrm{dgMod}_{\mathcal{O}}$ conc. in deg $[0,1]$

$$\rightsquigarrow \quad 0 \longrightarrow H^0K^\bullet \longrightarrow K^\bullet \underset{\underset{s}{\dashleftarrow}}{\longrightarrow} H^1K^\bullet[-1] \longrightarrow 0$$

$\Updownarrow$

decomp $\quad \bigoplus_{i=0,1} H^iK^\bullet[-i] \longrightarrow K^\bullet$

Define a presheaf of groupoids $\mathcal{G} : \mathrm{Open}(X)^{op} \longrightarrow \mathrm{Gpd}$ by

- $ob(\mathcal{G}(U)) = \{$section $s : H^1K^\bullet[-1] \longrightarrow K^\bullet$ of the projection$\}$
- $\mathrm{Hom}_{\mathcal{G}(U)}(s_1, s_2) = \left\{ \text{homotopy} \quad \begin{tikzcd} & K^0|_U \arrow[d, "d"] \\ H^1K^\bullet|_U \arrow[ur, "h"] \arrow[r, "s_2-s_1"'] & K^1|_U \end{tikzcd} \right\}$
- For $V \subset U$ restriction $\mathcal{G}(U) \longrightarrow \mathcal{G}(V)$

For $\mathcal{U} = \{U_\alpha\}$ open cover of U, consider the sheaf condition :

$$\mathcal{G}(U) \longrightarrow \underbrace{\left[\prod_\alpha \mathcal{G}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{G}(U_{\alpha\beta}) \to\!\!\!\!\!\rightrightarrows \prod \mathcal{G}(U_{\alpha\beta\gamma}) \right]}_{\text{holim}}$$

$=: \mathrm{Des}(\mathcal{U}, \mathcal{G})$

$\left\{\begin{array}{lll} \mathcal{G} : \text{prestack} & \text{if } \mathcal{G}(U) \longrightarrow \mathrm{Des}(\mathcal{U}, \mathcal{G}) & \text{fully faithful} \\ \quad\ \ \text{stack} & \text{of } \mathcal{G}(U) \longrightarrow \mathrm{Des}(\mathcal{U}, \mathcal{G}) & \text{equivalence} \end{array}\right.$ ("2-sheaf" of groupoids)

<u>Fact</u> • $\mathcal{G}$ defined above is a prestack

$\rightsquigarrow$ "stackification" $sc(K)(U) := \underset{\mathcal{U}:\text{ cover of } U}{\mathrm{hocolim}} \ \mathrm{Des}(\mathcal{U}, \mathcal{G})$

• If $K^\bullet$ is locally free, $sc(K)$ is a <u>gerbe</u>

i.e., it's "locally nonempty, connected" $\left(\begin{array}{l} \text{sheaf of "homotopy types"} \\ \text{with only } \pi_1 \neq 0 \end{array}\right.$

⋮