but I think it's the part
cited the most often

(preliminary part of)

"Serre-Tate local moduli"

by Katz

§1 "general" Serre-Tate theorem:

$$\begin{tikzcd}\text{deformation of abelian schemes} \arrow[d, leftrightarrow, "\wr"] \\ \text{deformation of the associated } p\text{-divisible groups}\end{tikzcd}$$

§2 "Serre-Tate coordinates" of the formal moduli of the deformations of <u>ordinary</u> abelian varieties

Not going to happen

sorry to disappoint you

§3 - §6 Coordinates interacts nicely with

Kodaira-Spencer map,

Gauss-Manin conn. on H_{dR}(formal moduli),

crystal structure (restated in many ways)

§1 "General" Serre-Tate theorem

R : comm. ring

Def $G : \mathrm{CRing}_R \longrightarrow \mathrm{Ab}$ is an R-group if it is an fppf-sheaf, i.e.

$\forall \{R \longrightarrow R_\alpha\}$ fppf cover

$R \to \prod R_\alpha$ faithfully flat, R_α : fin pres $/R$

$$G(R) \longrightarrow \prod G(R_\alpha) \rightrightarrows \prod G(R_\alpha \otimes_R R_\beta) : \text{exact}$$

$\rightsquigarrow$ abelian category

$$R\text{-Grp} = \mathrm{Shv}_{\mathrm{Ab}}^{\mathrm{fpqc}}(\mathrm{CRing}_R^{op})$$

Def • G : R-group scheme if $\exists X \in \mathrm{Sch}_R$

$$(\mathrm{CRing}_R \xrightarrow{G} \mathrm{Ab} \longrightarrow \mathrm{Set}) \cong \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(-), X)$$

• G : abelian scheme $/\mathrm{Spec}R$ if moreover

X : smooth proper with geom. conn. fibers

• G : finite flat R-group if

X : locally free of finite rank $/R$

($\Leftrightarrow$ finite flat lofp)

- G : <u>p-divisible group</u> $/R$ if

(Barsotti – Tate group)

$G[p^n] := \mathrm{Ker}(G \xrightarrow{p^n} G)$

$$\left\{\begin{array}{ll} (1) & G \xrightarrow[p]{} G : \text{epi} \\ (2) & \operatorname*{colim}_n G[p^n] \xrightarrow[\simeq]{} G \\ (3) & G[p^n] : \text{finite flat} \end{array}\right.$$

colim

$G_n = G[p^n]$

equivalently : $\{G_n,\ i_n : G_n \to G_{n+1}\}_{n \in \mathbb{N}}$

$$\left\{\begin{array}{l} \text{(i) } \exists\, h : \mathrm{Spec}\, R \to \mathbb{Z}_{\geq 1} \text{ locally const} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{“height”} \\ \qquad G_n : \text{finite flat of rk } p^{nh}, \\ \text{(ii) } 0 \to G_n \to G_{n+1} \xrightarrow{p^n} G_{n+1} : \text{exact} \end{array}\right.$$

<u>Def</u> AVar(R), BT(R) ⊂ R-Grp full subcat of abelian schemes / p-div groups

(stable under base change
⇝ functorial in R

$$\begin{array}{ccc} (-)[p^\infty] : \mathrm{AVar}(R) & \longrightarrow & \mathrm{BT}(R) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ A & \longmapsto & A[p^\infty] = \mathrm{colim}\, \underbrace{A[p^n]}_{\hookleftarrow \text{ fin flat}} \end{array}$$

Setting R : ring $\ni p$: nilpotent. ("I : p-complete" is enough)

$I \triangleleft R$ nilpotent, $R_0 = R/I$

(say $I^{\nu+1} = 0$)

Thm

$$\begin{tikzcd}
A \arrow[d, mapsto] & \mathrm{AVar}(R) \arrow[r, "{A[p^\infty]}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{BT}(R) \arrow[d] \\
A_0 \in & \mathrm{AVar}(R_0) \arrow[r, "{A_0[p^\infty]}"'] & \mathrm{BT}(R_0)
\end{tikzcd}$$

(2-categorical) is a pullback of categories

i.e.

$$\begin{tikzcd}
\mathrm{AVar}(R) \arrow[r, "\star"] & \mathrm{AVar}(R_0) \times_{\mathrm{BT}(R_0)} \mathrm{BT}(R) \\
A \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \left(\underset{\overset{\shortparallel}{A_{R_0}}}{A_0},\ A[p^\infty],\ A_0[p^\infty] \xrightarrow{\sim} A[p^\infty]_0\right) \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

is an equivalence.

§1.1 Preparation : $(N = p^n)$

$R : \mathbb{Z}/N\mathbb{Z}$ - alg, $N \geq 1$

$I \triangleleft R$ nilpotent, $I^{\nu+1} = 0$, $R_0 := R/I$.

Def $G_I \subset \hat{G} \subset G$ subfunctors defined by

$$\begin{tikzcd} G_I(A) := \mathrm{Ker}(G(A) \arrow[r] & G(A/IA)) \arrow[dd, "I \text{ nilpotent}"] \\ \cap & \\ \hat{G}(A) := \mathrm{Ker}(G(A) \arrow[r] & G(A^{red})) \end{tikzcd}$$

Remark $\mathrm{Spec}\, R_0 \longrightarrow \mathrm{Spec}\, R$

$$\begin{tikzcd} \rightsquigarrow \varphi : \mathrm{CRing}_{R_0}^{op\ \text{fppf}} \arrow[r, bend left=15, "\text{forget}"] \arrow[r, phantom, "\perp"] & \mathrm{CRing}_R^{op\ \text{fppf}} \arrow[l, bend left=15, "- \otimes_R R_0"] \end{tikzcd}$$

"Conti map" of fppf sites

$$\begin{tikzcd} R_0\text{-Grp} = \mathrm{Shv}_{Ab}^{fppf}(\mathrm{Spec}\, R_0) \arrow[r, bend right=15, "\varphi_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Shv}_{Ab}^{fppf}(\mathrm{Spec}\, R) = R\text{-Grp} \arrow[l, bend right=15, "\varphi^{-1}"'] \end{tikzcd}$$

unit is $\eta_G : G \longrightarrow \varphi_* \varphi^{-1} G \ : A \mapsto G(A/IA)$

So $G_I = \ker \eta_G$

Lemma G : (commutative) formal group $/R$

$$\Longrightarrow \quad N^{\nu} \text{ kills } G_I$$

proof choose coordinates smooth connected

$$G \cong \operatorname{Spf} R[\![x_1, \dots, x_n]\!]$$

group str : $R[\![x_1, \cdots, x_n]\!] \longrightarrow R[\![x_1, \dots, x_n]\!] \hat{\otimes} R[\![x_1, \dots, x_n]\!]$

$$x_i \longmapsto f_i(\vec{X}, \vec{Y})$$

$$\rightsquigarrow \begin{cases} \vec{f}(\vec{f}(\vec{X}, \vec{Y}), \vec{Z}) = \vec{f}(\vec{X}, \vec{f}(\vec{Y}, \vec{Z})) \\ \vec{f}(\vec{X}, 0) = \vec{X} = \vec{f}(0, \vec{X}) \\ \vec{f}(\vec{X}, \vec{Y}) = \vec{f}(\vec{Y}, \vec{X}) \end{cases}$$

$$\rightsquigarrow \vec{f}(\vec{X}, \vec{Y}) = \vec{X} + \vec{Y} + (\deg \geq 2)$$

$$[N]\vec{X} := \vec{f}(\vec{X}, \dots, \vec{X}) = \cancel{N} \vec{X} + \begin{pmatrix} \deg \geq 2 \\ \text{of } X_1, \dots, X_n \end{pmatrix}$$

$N = 0$ in R

$$[N^{\nu}]\vec{X} = (\deg \geq 2^{\nu})$$

WTS : $N^{\nu} = 0$ on $G_I(A)$, $\forall A \in \mathrm{CRing}_R$

$$G_I(A) = \left\{ f \;\middle|\; \begin{tikzcd} R[\![\vec{X}]\!] \arrow[r, "f"] \arrow[dr, "0"'] & A \arrow[d] \\ & A/IA \end{tikzcd} \quad \text{conti} \right\}$$

$f(X_i) \in IA$

$$\rightsquigarrow N^\nu f(X_i) = f(\overbrace{[N^\nu] X_i}^{\deg \geq 2^\nu \geq \nu+1}) \in I^{\nu+1}A = 0 \qquad \square$$

$$\sim f\Big(\overset{\shortparallel}{\sum} x^{\deg \geq 2^\nu}\Big) = \sum \underline{f(x)}^{\deg \geq 2^\nu} \in IA$$

Cor $G \in R\text{-Grp}$ s.t. $\hat{G}$: fppf locally covered by formal groups

$\Rightarrow N^\nu$ kills G_I

i.e. $\exists \{R \to R_\alpha\}$ fppf cover,

$$\begin{tikzcd} \mathrm{CRing}_R \arrow[r, "\hat{G}"] & \mathrm{Ab} \\ \mathrm{CRing}_{R_\alpha} \arrow[u] \arrow[ur, "\hat{G}_\alpha : \text{formal group}"'] & \end{tikzcd}$$

proof

- $\underset{\underset{G_I}{\shortparallel}}{(G_I)_I} \subset (\hat{G})_I \subset G_I \quad \rightsquigarrow (\hat{G})_I = G_I$

  $G_I(A/IA) = \ker\big(G(A/IA) \xrightarrow{\cong} G((A/IA)/\underset{\underset{0}{\shortparallel}}{(I \cdot A/IA)})\big)$ is zero

- $\hat{G}_I$ is covered by $(\hat{G}_\alpha)_I$: killed by N^ν.

$\rightsquigarrow \hat{G}_I$: killed by N^ν (by the sheaf property)

$\square$

Lemma A Let $G, H \in R\text{-Grp}$, suppose

(a) G is N-divisible (i.e., $G \xrightarrow{N} G$ epi)

(b) $\hat{H}$ is fppf-covered by formal groups

(c) H is formally smooth

(i.e., $J \subset A$ nilpotent $\Rightarrow H(A) \twoheadrightarrow H(A/J)$)

Recall $\varphi : R \twoheadrightarrow R/I = R_0$ "reduction mod I"

$$\rightsquigarrow \begin{tikzcd} R_0\text{-Grp} \arrow[r, bend right, "\varphi_*"'] \arrow[r, phantom, "\perp"] & R\text{-Grp} \arrow[l, bend right, "\varphi^{-1}"'] \end{tikzcd}$$

φ^{-1}: res to CAlg_{R_0}

$\varphi_* G_0 = G_0(- \otimes R_0)$

Then (Set $G_0 = \varphi^{-1} G$, $H_0 = \varphi^{-1} H$)

(1) $\mathrm{Hom}_{R\text{-Grp}}(G, H)$, $\mathrm{Hom}_{R_0\text{-Grp}}(G_0, H_0)$ have no N-torsion

(2) $\mathrm{Hom}_{R\text{-Grp}}(G, H) \xrightarrow{\overset{\varphi^{-1}_{G,H} =}{\theta}} \mathrm{Hom}_{R_0\text{-Grp}}(G_0, H_0)$: injective

(3) $\forall f_0 : G_0 \to H_0 \ \exists!\, g : G \to H$ s.t. $\theta(g) = N^{\nu} f_0$

(4) $f_0 \in \mathrm{Im}\,\theta \iff \underset{\substack{\uparrow \\ \text{in (3)}}}{g}|_{G[N^{\nu}]} = 0$

Proof (1) φ^{-1} preserves colim $\Rightarrow$ preserves epimorphisms

$\rightsquigarrow$ $N : G_0 \to G_0$ epi by (a)

$\rightsquigarrow$ $N \circlearrowright \mathrm{Hom}(G,H)\,,\ \mathrm{Hom}(G_0,H_0)$: mono.

(2)
$$\begin{tikzcd}
\mathrm{Hom}(G,H) \arrow[r, "\theta"] \arrow[dr, "\eta_H"'] & \mathrm{Hom}(\varphi^{-1}G,\varphi^{-1}H) \arrow[d, phantom, "\wr\|"] \\
& \mathrm{Hom}(G,\varphi_*\varphi^{-1}H)
\end{tikzcd}$$

$$\mathrm{Ker}\,\theta \cong \mathrm{Hom}(G, \underset{\substack{\| \\ H_I}}{\mathrm{Ker}\,\eta_H}) \circlearrowleft \underbrace{N^{\nu}}_{\substack{\uparrow \\ \text{mono} \\ \text{by (a)}}} \underset{\substack{\text{by Lem} \\ \& \\ \text{(b)}}}{= 0}$$

So $\mathrm{Ker}\,\theta = 0$.

(3) Uniqueness : by (2)

explicitly construct g

$$0 \to H_I(A) \to H(A) \longrightarrow \overbrace{H(A/IA) \longrightarrow 0}^{\text{by (c)}}$$

$$\begin{tikzcd}
\rightsquigarrow 0 \arrow[r] & H_I \arrow[r] \arrow[dr, "\underset{\text{by (b)}}{0}"'] & H \arrow[r, "\eta_H"'] \arrow[d, "N^{\nu}"'] & \varphi_*\varphi^{-1}H \arrow[r] \arrow[dl, dashed, "\exists !\, s"] & 0 \\
& & H & &
\end{tikzcd}$$

Define

$$\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "\eta_G"'] \arrow[dr, phantom, "\circlearrowright"] & H \arrow[dr, "\eta_H"] \arrow[dr, phantom, "\circlearrowright"', shift right=3] & \\
\underbrace{\varphi_*\varphi^{-1}G}_{G_0} \arrow[r, "\varphi_* f_0"'] & \underbrace{\varphi_*\varphi^{-1}H}_{H_0} \arrow[u, "s"'] \arrow[r, "N^\nu"'] & \varphi_*\varphi^{-1}H
\end{tikzcd}$$

$$\underbrace{\qquad\qquad}$$

$$\begin{tikzcd}
{} \arrow[r, two heads, "\eta_H"] \arrow[dr, "n"'] \arrow[dr, phantom, "\circlearrowright", shift right=2] & {} \arrow[d, "s"] & {} \arrow[dr, "n"] \arrow[d, phantom, "\circlearrowright"] & \\
 & {} & {} \arrow[r, "\eta_H"'] & {}
\end{tikzcd}$$

then

$$\begin{tikzcd}
\varphi^{-1}G \arrow[r, "\varphi^{-1}g = \theta(g)"] \arrow[d, "\varphi^{-1}\eta_G"] \arrow[dd, bend right=60, "\mathrm{id}"'] \arrow[dr, phantom, "\circlearrowright"] & \varphi^{-1}H \arrow[d, "\varphi^{-1}\eta_H"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
\varphi^{-1}\varphi_*\varphi^{-1}G \arrow[r, "\varphi^{-1}\varphi_* N^\nu f_0"'] \arrow[d, "\varepsilon_{\varphi^{-1}G}"'] \arrow[dr, phantom, "\circlearrowright"] & \varphi^{-1}\varphi_*\varphi^{-1}H \arrow[d, "\varepsilon_{\varphi^{-1}H}"] \\
\varphi^{-1}G \arrow[r, "N^\nu f_0"'] & \varphi^{-1}H
\end{tikzcd}$$

so $\theta(g) = N^\nu f_0$.

(4) θ : inj $\rightsquigarrow$ $\theta(f) = f_0 \iff N^\nu f = g$

$$\begin{tikzcd}
0 \arrow[r] & G[N^\nu] \arrow[r] \arrow[dr, "0"'] & G \arrow[r, "N^\nu"] \arrow[d, "g"] & G \arrow[r] \arrow[dl, dashed, "f"] & 0 \\
 & & H & &
\end{tikzcd}$$

$\rightsquigarrow \exists f \iff g|_{G[N^\nu]} = 0$ $\blacksquare$

§1.2 Proof of the theorem $\boxed{N = p^n}$

Step 1 show ☆ is fully faithful, i.e.

$f = g/p^{n\nu}$ $\forall A, B \in \mathrm{AVar}(R)$

$$\begin{tikzcd}
g \arrow[d, mapsto] \arrow[rrr, mapsto, bend left] & \mathrm{Hom}_{\mathrm{AVar}(R)}(A,B) \arrow[r] \arrow[d, "\theta_1"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{BT(R)}(A[p^\infty], B[p^\infty]) \arrow[d, "\theta_2"] & g[p^\infty] \arrow[d, mapsto] \\
p^{n\nu} f_0 \arrow[rrr, mapsto, bend right] & \mathrm{Hom}_{\mathrm{AVar}(R_0)}(A_0, B_0) \arrow[r] & \mathrm{Hom}_{BT(R_0)}(A_0[p^\infty], B_0[p^\infty]) & ⊛
\end{tikzcd}$$

ab. sch. & p-div gps satisfy (a)–(c) of Lemma A

$\rightsquigarrow$ θ_1, θ_2 : injective.

WTS $\forall f_0 : A_0 \to B_0$

[$f_0[p^\infty] \in \mathrm{Im}\,\theta_2 \Rightarrow f_0 \in \mathrm{Im}\,\theta_1$

$\exists!\, g : A \to B$ s.t. $g = p^{n\nu} f_0$.

Need $g|_{A[p^{n\nu}]} = 0$.

but $g[p^\infty] : A[p^\infty] \to B[p^\infty]$ lifts $p^{n\nu} \underbrace{f_0[p^\infty]}_{\in \mathrm{Im}\,\theta_2}$

$\Rightarrow g|_{A[p^{n\nu}]} = g[p^\infty]|_{A[p^{n\nu}]} = 0$ ☒

Step 2 essential surjectivity

$$\begin{tikzcd}
\mathrm{AVar}(R) \arrow[rr] \arrow[d] & B \arrow[ddl, mapsto] & BT(R) \arrow[d] \arrow[r, phantom, "\ni"] & G \arrow[d, mapsto] \\
\mathrm{AVar}(R_0) \arrow[rr] & & BT(R_0) & G_0 \\
A_0 \arrow[u, phantom, sloped, "\in"] \arrow[rr, mapsto] & & A_0[p^\infty] \arrow[ur, "\sim", "\varepsilon"'] &
\end{tikzcd}$$

Fact $\mathrm{AVar}(R) \longrightarrow \mathrm{AVar}(R_0)$: ess. surj.

Idea
- lifting of the underlying scheme exists by smoothness. $f : A \to \operatorname{Spec} R_0$
  (obstruction $\in \operatorname{Ext}^2(L_{A/R_0}, f^*I) = 0$)
- prove : abelian group str. lifts uniquely by deforming the graph of the structure maps

$$\rightsquigarrow \quad \exists\, B$$

$$\begin{tikzcd}
B \arrow[d, mapsto] & & & & \\
B_0 \arrow[r, "\sim", "\alpha_0"'] & A_0 & \longmapsto & B_0[p^\infty] \arrow[r, "\sim", "\alpha_0"'] & A_0[p^\infty] \arrow[r, "\sim"] & G_0
\end{tikzcd}$$

$$\begin{tikzcd}
BT(R) \arrow[d] \\
BT(R_0)
\end{tikzcd}$$

$$\begin{tikzcd}
B[p^\infty] \arrow[loop left, "p^{2n\nu}"] \arrow[r, bend left, "g"] & G \arrow[l, bend left, "h"] \arrow[loop right, "p^{2n\nu}"] \\
B_0[p^\infty] \arrow[loop below, "p^{2n\nu}"] \arrow[r, bend left, "p^{n\nu}\alpha_0"] & G_0 \arrow[l, bend left, "p^{n\nu}\alpha_0^{-1}"] \arrow[loop below, "p^{2n\nu}"] \arrow[u, squiggly, "\text{unique lift}"']
\end{tikzcd}$$

$\rightsquigarrow$ g, h : isogeny

Consider

$$\begin{tikzcd}
0 \arrow[r] & K = \mathrm{Ker}\, g \arrow[r] \arrow[d] & B[p^\infty] \arrow[r, "g"] \arrow[d, equal] & G \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & B[p^{2n\nu}] \arrow[r] & B[p^\infty] \arrow[r, "p^{2n\nu}"'] & B[p^\infty] \arrow[r] & 0
\end{tikzcd}$$

g : flat by "fiberwise criterion"

$$\begin{tikzcd}
B[p^\infty] \arrow[rr] \arrow[dr, "\text{flat}"'] & & G \arrow[dl] & {} \arrow[r, squiggly, "\text{fiber}"'] & {} & B[p^\infty]_s \arrow[rr, "p^{n\nu}\circ\text{isom : flat}"] \arrow[dr] & & G_s \arrow[dl] \\
& \mathrm{Spec}\, R \arrow[dr, squiggly] & & & & & \{s\} & \\
& & B_0[p^\infty] \arrow[rr] \arrow[dr] & & G_0 \arrow[dl] \arrow[urr, squiggly] & & & \\
& & & \mathrm{Spec}\, R_0 & & & &
\end{tikzcd}$$

finiteness ?

<u>lofp ?</u>

$\rightsquigarrow$ $K \hookrightarrow B[p^{2n\nu}]$ finite flat subgroup

$$A = B/K \in \mathrm{AVar}(R)$$

Then

$$\begin{tikzcd}
 & & & \mathrm{AVar} & & & & \mathrm{BT} & \\
R & & K \arrow[r] & B \arrow[r] \arrow[dd, mapsto] & A \arrow[rr, mapsto] & & K \arrow[r] & B[p^\infty] \arrow[r] \arrow[dr, "\underset{\sim}{g}"'] \arrow[dd, mapsto] & A[p^\infty] \arrow[d, "\cong"] \\
 & & & & & & & & G \\
R_0 & & B_0[p^{n\nu}] \arrow[r] & B_0 \arrow[r] \arrow[d] & A \underset{R}{\otimes} R_0 \arrow[rr, mapsto] \arrow[dr, "\sim"] & & B_0[N^\nu] \arrow[r] & B_0[p^\infty] \arrow[r] \arrow[dr, "\underbrace{p^{n\nu}\cdot\alpha_0[p^\infty]}_{\text{compatible}}"'] & A_0[p^\infty] \arrow[d, "\cong"] \\
 & & & B_0/B_0[p^{n\nu}] \arrow[r, "\sim", "p^{n\nu}"'] & B_0 \arrow[r, "\sim", "\alpha_0"'] & A_0 & & & G_0
\end{tikzcd}$$

i.e. A lifts $(A_0, G, \alpha : A_0[p^\infty] \to G_0)$ $\blacksquare$

§2 Serre-Tate coordinates for deformations of ordinary abelian varieties

Notations

- $A \in \mathrm{AVar}(k)$, k : $\overset{\text{char } p}{\text{field}}$, $g = \dim A$
- $(-)^t$: the $\overset{\text{(cartier)}}{\text{dual}}$ of ab sch / fin flat gp / p-div gp
- $T_pA(k) := \varprojlim_n A(k)[p^n]$: the Tate module (a $\mathbb{Z}_p$-module)
- k : field $\rightsquigarrow$ $\underset{\substack{\rotatebox{90}{$\in$} \\ (R,\mathfrak{m})}}{\mathrm{Art}_k}$: the cat of augmented artin local k-alg
- $$\begin{tikzcd} \widehat{\mathcal{M}}_{A/k} : \mathrm{Art}_k \arrow[r] & \mathrm{Cat} \arrow[r, "\pi_0"] & \mathrm{Set} \\ R \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & \{A\} \times_{\mathrm{AVar}(k)} \mathrm{AVar}(R) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & \end{tikzcd}$$

"the formal moduli space"

$\mathcal{A}$

$\downarrow$

known to be prorepresentable by $\mathrm{Spf}\, \mathcal{R}$,

$$\mathcal{R} \simeq W(k)[[t_{ij} \mid 1 \leq i,j \leq g]]$$

Thm Setting: k : alg closed field $\supset \mathbb{F}_p$

$A, B \in \mathrm{AVar}(k)$ ordinary

(1) $\exists$ natural isom

$$\begin{tikzcd}
\widehat{\mathcal{M}}_{A/k}(R) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{Z}_p}\Big(T_pA(k) \underset{\mathbb{Z}_p}{\otimes} T_pA^t(k), \overset{\overset{(\widehat{\mathbb{G}}_m)^{g^2}}{\cong}}{} \overset{\overset{1+\mathfrak{m}}{\parallel}}{\widehat{\mathbb{G}}_m(R)}\Big) \\
\mathbb{A}/R \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & q(\mathbb{A}/R\,;-,-) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

of functors $\mathrm{Art}_k \longrightarrow \mathrm{Set}$

(2) Compatible with the duality :

$$\begin{tikzcd}
\widehat{\mathcal{M}}_{A/k}(R) \arrow[r, "\cong"] \arrow[d, "(-)^t"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k)\otimes T_pA^t(k), \widehat{\mathbb{G}}_m(R)) \arrow[d, "\cong"] \\
\widehat{\mathcal{M}}_{A^t/k}(R) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k)\otimes T_pA^{tt}(k), \widehat{\mathbb{G}}_m(R))
\end{tikzcd}$$

(3) $\mathbb{A}, \mathbb{B}$: lifts of A, B, then $A \xrightarrow{f} B$ lifts to $\exists\, \mathbb{A} \xrightarrow{\mathbf{f}} \mathbb{B}$

iff

$$\begin{tikzcd}
& T_pA(k)\otimes T_pA^t(k) \arrow[dr, "q(\mathbb{A}/R\,;-,-)"] & \\
T_pA(k)\otimes T_pB^t(k) \arrow[ur, "\mathrm{id}\otimes f^t"] \arrow[dr, "f\otimes \mathrm{id}"'] & \circlearrowright & \widehat{\mathbb{G}}_m(R) \\
& T_pB(k)\otimes T_pB^t(k) \arrow[ur, "q(\mathbb{B}/R\,;-,-)"'] &
\end{tikzcd}$$

Some definitions & preliminaries (R: artin local)

- X: fin flat R-group scheme is
  - **étale** if $X \to \operatorname{Spec} R$: étale
  - **of multiplicative type** if étale-locally (on $\operatorname{Spec} R$) isomorphic to $\exists \bigoplus_i \mu_{n_i}$

dual

- X: p-divisible group $/_R$ is
  - **étale** if $X[p^n]$: étale $\forall n$

    connected-étale seq ← canonically splits if R: perfect field

$$0 \to X_{conn} \to X \to X_{ét} \to 0$$

connected component of the unit ↗ (at X_{conn}) ; ↑ maximal étale quotient (at $X_{ét}$)

  - **connected** if $X_{ét} = 0$
  - **toric** (toroidal) if $X[p^n]$: multiplicative type $\iff X^t$: étale

$$0 \to X^t{}_{conn} \to X^t \to X^t{}_{ét} \to 0$$

dual ⤳

$$0 \to \underbrace{(X^t_{ét})^t}_{= X_{tor}} \to X \to (X^t_{conn})^t \to 0$$

$$X_{tor} \hookrightarrow X_{conn}.$$

these are representable by formal groups $/R$

- ordinary if $X_{tor} \xrightarrow[\cong]{} X_{conn}$, i.e.

$$0 \to X_{tor} \to X \to X_{ét} \to 0$$

- A: abelian variety

$\rightsquigarrow$ $A[p^\infty]_{conn} \cong \hat{A}$: the formal completion along the zero section $\operatorname{Spec} R \to A$

$\left(\hat{A}(R) = \ker(A(R) \to A(R^{red})) \text{ for } R\text{: artinian local} \right)$

- A is ordinary if $A[p^\infty]$ is ordinary

$\iff A[p^n] \cong (\mathbb{Z}/p^n\mathbb{Z})^g$

$\iff T_pA(k) \cong \mathbb{Z}_p{}^g$

typical examples

<u>fin flat X</u>

X	unip	mult
$X \backslash X^t$	conn	ét
conn	α_p	μ_{p^n}
ét	$\mathbb{Z}/p^n\mathbb{Z}$	$\mathbb{Z}/\ell$

μ_ℓ

dual

$\alpha_p = \ker(Fr : \mathbb{G}_a \to \mathbb{G}_a)$

$\mu_{p^n} \rightsquigarrow \mu_{p^\infty}$

$$\mathbb{Z}/p^n\mathbb{Z} = \frac{1}{p^n}\mathbb{Z}/\mathbb{Z} \rightsquigarrow \underline{\mathbb{Q}_p/\mathbb{Z}_p}$$

Construction / Proof of (1)

$$\begin{tikzcd}
A \arrow[d, mapsto] & \in & \pi_0\left(\{A\} \times_{\mathrm{AVar}(k)} \mathrm{AVar}(R)\right) \arrow[d, "\cong"', "\text{by } \S 1"] \\
A[p^\infty] \arrow[d, mapsto] & \in & \pi_0\left(\{A[p^\infty]\} \times_{\mathrm{BT}(k)} \mathrm{BT}(R)\right) \arrow[d, "\cong"', "①"] \\
0 \to \boxed{\text{tor}} \to A[p^\infty] \to \boxed{\text{ét}} \to 0 & \in & \mathrm{Ext}^1_{R\text{-Grp}}\left(\underbrace{T_pA(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p}_{\boxed{\text{ét}}}, \underbrace{\mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k),\hat{\mathbb{G}}_m)}_{\boxed{\text{tor}}}\right) \arrow[d, shift left, "\cong", "②"'] \\
 & & \mathrm{Hom}_{R\text{-Grp}}\left(T_pA(k), \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k),\hat{\mathbb{G}}_m)\right) \arrow[u, shift left, "\cong"] \arrow[d, "\cong"', "\text{R-points \& tensor-hom}"] \\
 & & \mathrm{Hom}_{\mathbb{Z}_p}\left(T_pA(k)\otimes T_pA^t(k), \hat{\mathbb{G}}_m(R)\right)
\end{tikzcd}$$

$$\begin{tikzcd}
\boxed{\text{tor}} \arrow[r] & A[p^\infty] \\
T_pA(k) \arrow[u, "\varphi_A"] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & T_pA(k)\otimes\mathbb{Q}_p \arrow[u]
\end{tikzcd}$$

$\varphi_{A/R} \in$

① Classifying the lift $A[p^\infty]/_R$ of $A[p^\infty]/_k$

$\underline{\underline{n \gg 0}}$ so that p^n kills $\hat{A}$

$$\begin{tikzcd}
\rightsquigarrow & 0 \arrow[r] & \hat{A} \arrow[r] \arrow[d] & A[p^n] \arrow[r] \arrow[d] & A(k)[p^n] \arrow[d] & \\
& 0 \arrow[r] & \hat{A} \arrow[r] \arrow[d, "0"] & A \arrow[r] \arrow[d, "p^n"] & A(k) \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & \hat{A} \arrow[r] \arrow[d] & A \arrow[r] & A(k) \arrow[r] & 0 \\
& & \hat{A} & & &
\end{tikzcd}$$

vary n

$$\begin{tikzcd}
\rightsquigarrow & 0 \arrow[r] & \hat{A} \arrow[r] \arrow[d, "\mathrm{id}"'] & A[p^n] \arrow[r] \arrow[d, hook] & A(k)[p^n] \arrow[r] \arrow[d, hook] & \hat{A} \arrow[d, "p"] \\
& 0 \arrow[r] & \hat{A} \arrow[r] & A[p^{n+1}] \arrow[r] & A(k)[p^{n+1}] \arrow[r] & \hat{A}
\end{tikzcd}$$

$$\Big\downarrow \text{colim}$$

$$0 \to \hat{A} \to A[p^\infty] \to T_pA(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

this reduces to the (canonically split) conn-ét seq $/k$:

$$0 \to \hat{A} \to A[p^\infty] \to T_pA(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

A: ordinary $\iff$ $\hat{A} \simeq \left(\left(A[p^\infty]^t\right)_{ét}\right)^t$

did I use alg closed ?

$$\rightsquigarrow \begin{cases} \hat{A}[p^n] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}\left(A^t(k)[p^n], \mu_{p^n}\right) \\ \hat{A} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}\left(T_pA^t(k), \hat{\mathbb{G}}_m\right) \end{cases}$$

conti

note: $\hat{\mathbb{G}}_m(R) \simeq 1+m \simeq \mu_{p^\infty}(R)$

f.f. groups of multiplicative type / toroidal p-div gps $/k$

lifts uniquely to R

(probably b/c taking the dual, by $Ét_k \simeq Ét_R$?)

$$\rightsquigarrow \begin{cases} \hat{A}[p^n] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}\left(A^t(k)[p^n], \mu_{p^n}\right) \\ \hat{A} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}\left(T_pA^t(k), \hat{\mathbb{G}}_m\right) \end{cases}$$

$$\begin{tikzcd} \rightsquigarrow & 0 \arrow[r] & \hat{A} \arrow[r] & A[p^\infty] \arrow[r] \arrow[d] & T_pA(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \arrow[r] & 0 \\ & 0 \arrow[r] & \hat{A} \arrow[r] & A[p^\infty] \arrow[r] & T_pA(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \arrow[r] & 0 \end{tikzcd}$$

predetermined

toric

étale

classifying $A[p^\infty]$ $\iff$ classifying ext

② In general

$0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ abelian

$\left(X \to Y \to Z \quad \text{distinguished triangle in } D(\mathcal{A}) \right)$

$$\rightsquigarrow \mathrm{Hom}(Y,D) \underset{\textcircled{a}}{\longrightarrow} \mathrm{Hom}(X,D) \underset{\textcircled{b}}{\longrightarrow} \mathrm{Ext}^1(Z,D) \underset{\textcircled{c}}{\longrightarrow} \mathrm{Ext}^1(Y,D)$$

- If ⓐ, ⓒ $= 0$, then ⓑ bijective

explicitly

$$\begin{tikzcd}
\Omega Y \arrow[r] & \Omega Z \arrow[r] \arrow[dr] & X \arrow[r] \arrow[d, dashed, "\exists !"] & Y & : \text{fiber seq.} \\
\mathrm{Ext}^1(Z,D) \ni \arrow[d, leftrightsquigarrow] & & D & \in \mathrm{Hom}(X,D) \\
\Omega Z \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d, equal] \\
\Omega Z \arrow[r, "f"] & D \arrow[r] & \mathrm{cofib}(f) \arrow[r] & Z
\end{tikzcd}$$

Our case: $\mathcal{A} = R\text{-Grp}$

$$0 \to \underset{X}{T_pA(k)} \to \underset{Y}{T_pA(k) \otimes \mathbb{Q}_p} \to \underset{Z}{T_pA(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \to 0$$

$$D = \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \hat{\mathbb{G}}_m) \simeq \hat{A}$$

- $\mathrm{Hom}_{R\text{-Grp}}(T_pA(k)\otimes\mathbb{Q}_p\,,\ \hat{A}) = 0 \quad \Rightarrow ⓐ = 0$

mult. by p is surjective

formal group ⇝ killed by p-power

- $\mathrm{Ext}^1_{R\text{-Gp}}(T_pA(k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\,,\ \hat{A}) \longrightarrow \mathrm{Ext}^1_{R\text{-Gp}}(T_pA(k), \hat{A})$ injective

$\Rightarrow ⓒ = 0$

<u>sketch</u> $T_pA(k)\cong\mathbb{Z}_p^{\oplus g}$, $\hat{A}\cong\mu_{p^\infty}^{\oplus g}$ ⇝ enough to show

$$\begin{tikzcd}
\mathrm{Ext}^1_{R\text{-Grp}}(\mathbb{Q}_p, \mu_{p^\infty}) \arrow[r] \arrow[d, "①"] & \mathrm{Ext}^1_{R\text{-Grp}}(\mathbb{Z}_p, \mu_{p^\infty}) \\
\varprojlim \mathrm{Ext}^1_{R\text{-gp}}(\mathbb{Z}_p, \mu_{p^\infty}) \arrow[ru, "② \ pr"'] &
\end{tikzcd}$$

: inj

$\mathbb{Q}_p \cong \varinjlim \mathbb{Z}_p$

$\uparrow$ mult by p

$$\begin{tikzcd}
\mathbb{Z}_p \arrow[r, "1"] \arrow[d, equal] & \mathbb{Z}_p \arrow[r] \arrow[d, "p"] & 0 \arrow[d] \\
\mathbb{Z}_p \arrow[r, "p"] \arrow[d, equal] & \mathbb{Z}_p \arrow[r] \arrow[d, "p"] & \mathbb{Z}/p \arrow[d] \\
\mathbb{Z}_p \arrow[r, "p^2"] \arrow[d] & \mathbb{Z}_p \arrow[r] \arrow[d] & \mathbb{Z}/p^2 \arrow[d] \\
\vdots & \vdots & \vdots \\
\mathbb{Z}_p \arrow[r] & \mathbb{Q}_p \arrow[r] & \mathbb{Q}_p/\mathbb{Z}_p
\end{tikzcd}$$

① : inj

$$\begin{tikzcd}
\pi_{-1}\varprojlim \mathrm{Map}(\mathbb{Z}_p, \mu_{p^\infty}) \arrow[r] & \varprojlim \pi_{-1}\mathrm{Map}(\mathbb{Z}_p, \mu_{p^\infty}) \\
\mathrm{Ker} = \varprojlim^1 \underline{\pi_0\mathrm{Map}(\mathbb{Z}_p, \mu_{p^\infty})} \arrow[u] &
\end{tikzcd}$$

Mittag-Leffler condition ✓

② : inj

<u>follows from</u> $H^1_{ét}(\mathrm{Spec}\,R, \mu_{p^\infty}) = 0$

$\leftarrow$ not sure how

So

$$\begin{tikzcd}
\underset{\sim}{\Omega}\left(T_pA(k)\otimes \mathbb{Q}_p/\mathbb{Z}_p\right) \arrow[r, "{\hat{A}[p^\infty]}"] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}_p}\left(T_pA^t(k), \hat{\mathbb{G}}_m\right) \\
T_pA(k) \arrow[ur, dashed, "\exists ! \varphi_{A/R}"'] &
\end{tikzcd}$$

$[-1] \nearrow$

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

$$\underline{\otimes T_pA(k)}$$

$\left((1) \boxed{k} \right)$

Remark

More explicitly, $\varphi_{A/R}$ is given as follows:

take $n \gg 0$ so that $m^{n+1} = 0$

($p \in m$, so p^n kills $\hat{A}$)

$$\begin{tikzcd}
 & & & T_pA(k) \arrow[d] & \\
 & \hat{A} \arrow[r] \arrow[d] & A[p^n] \arrow[r] \arrow[d] & A(k)[p^n] \arrow[d] & \\
0 \arrow[r] & \hat{A} \arrow[r] \arrow[d, "0"] & A \arrow[r] \arrow[d, "p^n"] & A(k) \arrow[r] \arrow[d, "p^n"] & 0 \\
0 \arrow[r] & \hat{A} \arrow[r] \arrow[d] & A \arrow[r] & A(k) & \\
 & \hat{A} & & &
\end{tikzcd}$$

Proof

$$\begin{tikzcd}
 & & & & & & & A(k)[p^n] \arrow[dl] \arrow[dd, "id"] \\
 & & T_pA(k) \arrow[rr] \arrow[dl, "p^n"'] \arrow[dd] & & T_pA(k)\otimes\mathbb{Q}_p \arrow[rr] \arrow[dl, "p^n"] \arrow[dd] & & T_pA(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \arrow[dl, "p^n"] \arrow[dd] & \\
 & T_pA(k) \arrow[rr] \arrow[dl] \arrow[dd, "\varphi"] & & T_pA(k)\otimes\mathbb{Q}_p \arrow[rr] \arrow[dd] & & T_pA(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \arrow[dd, "\|"'] & & A(k)[p^n] \arrow[dl] \\
A(k)[p^n] \arrow[dd, "\varphi'"'] & & \hat{A} \arrow[rr] \arrow[dl, "p^n=0"] & & A[p^\infty] \arrow[rr] \arrow[dl, "p^n"] & & T_pA(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p \arrow[dl, "p^n"] & \\
 & \hat{A} \arrow[rr] \arrow[dl, "id"] & & A[p^\infty] \arrow[rr] & & T_pA(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p & & \\
\hat{A} & & & & & & &
\end{tikzcd}$$

$$\Big\downarrow$$

$$\begin{tikzcd}
A(k)[p^n] \arrow[d, "\varphi'"'] & A(k)[p^n] \arrow[l, "id"'] \arrow[d, "id"] \\
\hat{A} & A(k)[p^n] \arrow[l]
\end{tikzcd}$$

i.e. φ' = (the connecting hom)

Proof of (3)

$$\begin{tikzcd}
A \arrow[r, dashed, "f"'] \arrow[dd] & B \arrow[dd] & & A[p^\infty] \arrow[r, dashed, "{f[p^\infty]}"] \arrow[dd] & B[p^\infty] \arrow[dd] \\
& {} \arrow[rr, leftrightsquigarrow, "\text{Serre-Tate}"] & & {} & \\
A \arrow[r, "f"'] & B & & A[p^\infty] \arrow[r, "{f[p^\infty]}"'] & B[p^\infty]
\end{tikzcd}$$

by construction of (1), $f[p^\infty]$ is a morphism filling

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \hat{\mathbb{G}}_m) \arrow[r] \arrow[d, "{\mathrm{Hom}(f^t, id)}"'] & A[p^\infty] \arrow[r] \arrow[d, dashed, "{f[p^\infty]}"] & T_pA(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \arrow[d, "f \otimes id"] \\
\mathrm{Hom}_{\mathbb{Z}_p}(T_pB^t(k), \hat{\mathbb{G}}_m) \arrow[r] & B[p^\infty] \arrow[r] & T_pB(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p
\end{tikzcd}$$

pre-determined by the unique liftability of toroidal / étale groups to nilpotent thickening rotating the triangle :

$$\begin{tikzcd}
\Omega(T_pA(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p) \arrow[r, "A[p^\infty]"'] \arrow[d, "f\otimes id"] & \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \hat{\mathbb{G}}_m) \arrow[r] \arrow[d, "\mathrm{Hom}(f^t, id)"] & A[p^\infty] \arrow[d, dashed, "f[p^\infty]"] \\
\Omega(T_pB(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p) \arrow[r, "B[p^\infty]"'] & \mathrm{Hom}_{\mathbb{Z}_p}(T_pB^t(k), \hat{\mathbb{G}}_m) \arrow[r] & B[p^\infty]
\end{tikzcd}$$

$\exists f[p^\infty] \iff$ left square commutes

$$\begin{tikzcd}
& T_pA(k) \arrow[ddd, "f"] \arrow[dr, dashed, "\exists!\, \varphi_{A/R}"] & & \\
\Omega(T_pA(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p) \arrow[ur] \arrow[rr] \arrow[d, "f\otimes id"] & & \mathrm{Hom}_{\mathbb{Z}_p}(T_pA^t(k), \hat{\mathbb{G}}_m) \arrow[r] \arrow[d, "\mathrm{Hom}(f^t, id)"] & A[p^\infty] \arrow[d, dashed, "f[p^\infty]"] \\
\Omega(T_pB(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p) \arrow[rr] \arrow[dr] & & \mathrm{Hom}_{\mathbb{Z}_p}(T_pB^t(k), \hat{\mathbb{G}}_m) \arrow[r] & B[p^\infty] \\
& T_pB(k) \arrow[ur, dashed, "\exists!\, \varphi_{B/R}"'] & &
\end{tikzcd}$$

$\iff$ blue square commutes

$\iff$

$$\begin{tikzcd}
& T_pA(k)\otimes T_pA^t(k) \arrow[dr, "e(A/R;-,-)"] & \\
T_pA(k)\otimes T_pB^t(k) \arrow[ur, "f\otimes id"] \arrow[dr, "id\otimes f^t"'] & \circlearrowright & \mathbb{G}_m \\
& T_pB(k)\otimes T_pB^t(k) \arrow[ur, "e(B/R;-,-)"'] &
\end{tikzcd}$$

Proof of (2) is technical and unenlightening contrary to its formal appearance ...

§ 3

$$\hat{\mathcal{M}}_{A/k} : \mathrm{Art}_k \longrightarrow \mathrm{Set}$$

representable by a complete local alg $\overset{\substack{\text{Canonically an} \\ W(k)\text{-alg} \\ \downarrow}}{(\mathcal{R}, m)}$

$$\hat{\mathcal{M}}_{A/k} \simeq \mathrm{Spf}\,\mathcal{R} \multimap \underline{\text{formal } W(k)\text{-group}}$$

(e.g. by Schlessinger's representability thm)

$$\left(\begin{array}{l} \text{Since } \hat{\mathcal{M}}_{A/k} \xrightarrow[q]{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), \hat{\mathbb{G}}_m) \\ \qquad\qquad\qquad\qquad\qquad\qquad \wr\mid \\ \qquad\qquad\qquad\qquad\qquad\qquad \hat{\mathbb{G}}_m^{g^2} \\ \text{by picking } \mathbb{Z}_p\text{-basis } \alpha_1, \dots, \alpha_g \text{ of } T_pA \\ \qquad\qquad\qquad\qquad\quad \beta_1, \dots, \beta_g \text{ of } T_pA^t \\ \text{and setting} \\ \qquad T_{ij} = q(\alpha_i, \alpha_j) - 1 \in \mathcal{R} \\ \rightsquigarrow \; W(k)[\![T_{ij}]\!] \xrightarrow{\sim} \mathcal{R} \end{array} \right)$$

passing to the limit

$$\underline{\hat{\mathcal{M}}_{A/k} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes T_pA^t(k), \hat{\mathbb{G}}_m)}$$

$$q : T_pA(k) \otimes T_pA^t(k) \xrightarrow[\simeq]{} \mathrm{Hom}_{W(k)\text{-Grp}}(\hat{\mathcal{M}}, \hat{\mathbb{G}}_m)$$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} R \arrow[r] & \operatorname{Spf} \mathcal{R}
\end{tikzcd}$$

the universal deformation

$$\begin{tikzcd}
R & \mathcal{R} \arrow[l, "f_{\mathbb{A}/R}"'] \\
q(\mathbb{A}/R, \alpha, \beta) & q(\alpha, \beta) \arrow[l, mapsto]
\end{tikzcd}$$

$$KS : \underline{\omega}_{A/R} \longrightarrow \mathrm{Lie}(A^t/R) \underset{R}{\otimes} \Omega^1_{R/W}$$

$$\left(\xrightarrow{\text{limit}} \quad KS : \underline{\omega}_{\mathcal{A}/\mathcal{R}} \longrightarrow \mathrm{Lie}(\mathcal{A}^t/\mathcal{R}) \underset{\mathcal{R}}{\otimes} \Omega^1_{\mathcal{R}/W} \right.$$

$\hookleftarrow$ conti 1-forms

The main theorem

$$\begin{tikzcd}
(\alpha, \beta) \in \arrow[d, mapsto] & T_pA^{tt}(k) \otimes T_pA(k) \arrow[rr, "q"] \arrow[d] & & \mathrm{Hom}_{W\text{-}Grp}(\hat{\mathcal{U}}, \hat{\mathbb{G}}_m) \arrow[dd, "d\log"] \\
(\omega(\alpha), \omega(\beta)) \in & \underline{\omega}_{\mathcal{A}^t/\mathcal{R}} \otimes \underline{\omega}_{\mathcal{A}/\mathcal{R}} \arrow[d, "id \times KS"] & \circlearrowright & \\
& \underline{\omega}_{\mathcal{A}^t/\mathcal{R}} \otimes \mathrm{Lie}(\mathcal{A}^t/\mathcal{R}) \otimes \Omega^1_{\mathcal{R}/W} \arrow[rr, "\text{pairing} \otimes id"'] & & \Omega^1_{\mathcal{R}/W}
\end{tikzcd}$$

where $\omega(\beta)$ is (the limit of)

$$\begin{tikzcd}
T_pA^t(k) \arrow[r, "\sim"] \arrow[ddr, bend right, "\omega"'] & \mathrm{Hom}_{R\text{-}Grp}(\hat{A}, \hat{\mathbb{G}}_m) \arrow[d, "\mathrm{Lie}"] \\
& \mathrm{Hom}_{R\text{-}Grp}(\mathrm{Lie}(A/R), \mathbb{G}_a) \arrow[d, equal] \\
& \underline{\omega}_{A/R}
\end{tikzcd}$$

Restatement (§4)

Hodge-de Rhan

$$0 \to \underline{\omega}_{\mathcal{A}/\mathcal{R}} \longrightarrow H^1_{dR}(\mathcal{A}/\mathcal{R}) \longrightarrow \mathrm{Lie}(\mathcal{A}^t/\mathcal{R}) \to 0$$

Gauss-Manin connection

$$\nabla : H_{\mathcal{A}} \to H_{\mathcal{A}} \otimes \Omega^1_{\mathcal{R}/W(k)}$$