

(preliminary part of) but I think it's the part cited the most often

# "Serre-Tate local moduli"

by Katz

§ 1 "general" Serre-Tate theorem:

deformation of abelian schemes



deformation of the associated  $p$ -divisible groups

§ 2 "Serre-Tate coordinates" of the formal moduli of the deformations of ordinary abelian varieties

Not going to happen  
to sorry  
disappoint you

§ 3 - § 6 Coordinates interacts nicely with

Kodaira-Spencer map,

Gauss-Manin conn. on  $H_{\text{DR}}$ (formal moduli),  
crystal structure (restated in many ways)

# § 1 "General" Serre - Tate theorem

$R$ : Comm. ring

Def  $G: \text{CRing}_R \rightarrow \text{Ab}$  is an  $R$ -group if it is an fppf-sheaf, i.e.

$$\left[ \begin{array}{l} \forall \{R \rightarrow R_\alpha\} \text{ fppf cover} \\ G(R) \rightarrow \prod G(R_\alpha) \rightrightarrows \prod G(R_\alpha \otimes_R R_\beta) : \text{exact} \end{array} \right. \begin{array}{l} R \rightarrow \prod R_\alpha \\ \text{faithfully flat} \\ R_\alpha: \text{fin pres } / R \end{array}$$

$\rightsquigarrow$  abelian category

$$R\text{-Grp} = \text{Shv}_{\text{Ab}}^{\text{fppf}}(\text{CRing}_R^{\text{op}})$$

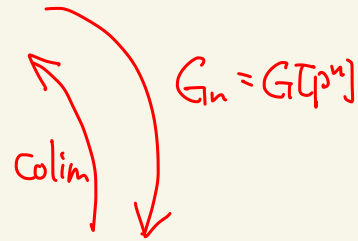
Def.  $G: R$ -group scheme if  $\exists X \in \text{Sch}_R$   
 $(\text{CRing}_R \xrightarrow{G} \text{Ab} \rightarrow \text{Set}) \cong \text{Hom}_{\text{Sch}}(\text{Spec}(-), X)$

- $G: \underline{\text{abelian scheme}} / \text{Spec } R$  if moreover  
 $X$ : smooth proper with geom. conn. fibers
- $G: \underline{\text{finite flat}} R$ -group if  
 $X$ : locally free of finite rank /  $R$   
( $\Leftrightarrow$  finite flat (lfp))

•  $G$  :  $p$ -divisible group /  $R$  if  
(Barsotti - Tate group)

$$G[p^n] := \ker(G \xrightarrow{p^n} G)$$

- (1)  $G \xrightarrow{p} G$  : epi  
 (2)  $\text{colim}_n G[p^n] \xrightarrow{\cong} G$   
 (3)  $G[p^n]$  : finite flat



equivalently :  $\{G_n, \lambda_n : G_n \rightarrow G_{n+1}\}_{n \in \mathbb{N}}$

- (i)  $\exists h : \text{Spec } R \rightarrow \mathbb{Z}_{\geq 1}$  locally const  
 "height"  
 $G_n$  : finite flat of rk  $p^{nh}$ ,  
 (ii)  $0 \rightarrow G_n \rightarrow G_{n+1} \xrightarrow{p^n} G_{n+1} : \text{exact}$

Def  $\text{AVar}(R), \text{BT}(R) \subset R\text{-Grp}$  full subcat of  
abelian schemes /  $p$ -div groups

stable under base change  
 functorial in  $R$

$$(-)[p^\infty] : \text{AVar}(R) \rightarrow \text{BT}(R)$$

$$A \longmapsto A[p^\infty] = \text{colim}_n A[p^n]$$

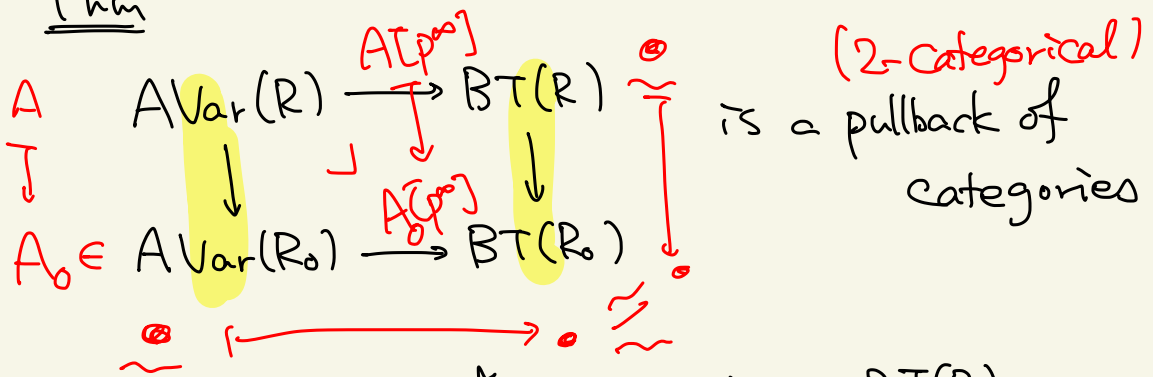
fin flat

Setting  $R$ : ring  $\ni p$ : nilpotent. ← ("I: p-Complete" is enough)

$I \triangleleft R$  nilpotent,  $R_0 = R/I$

(say  $I^{v+1} = 0$ )

Thm



(2-categorical)

is a pullback of categories

i.e.  $A \text{Var}(R) \xrightarrow{\star} A \text{Var}(R_0) \times_{BT(R_0)} BT(R)$

$$A \hookrightarrow (A_0, A[p^\infty], A_0[p^\infty] \xrightarrow{\cong} A[p^\infty]_0)$$

ii  
 $A_{R_0}$

is an equivalence.

§1.1 Preparation :  $(N = p^n)$

$R: \mathbb{Z}/N\mathbb{Z}$  - alg,  $N \geq 1$

$I \triangleleft R$  nilpotent,  $I^{N+1} = 0$ ,  $R_0 := R/I$ .

Def  $G_I \subset \widehat{G} \subset G$  subfunctors defined by

$$G_I(A) := \text{Ker}(G(A) \rightarrow G(A/IA))$$

$$\widehat{G}(A) := \text{Ker}(G(A) \rightarrow G(A^{\text{red}}))$$

$\downarrow I \text{ nilpotent}$   
 $\uparrow$

Remark  $\text{Spec } R_0 \rightarrow \text{Spec } R$

$$\rightsquigarrow \varphi: \text{CRing}_{R_0}^{\text{fppf}} \begin{array}{c} \xrightarrow{\text{forget}} \\ \perp \\ \xleftarrow{- \otimes_R R_0} \end{array} \text{CRing}_R^{\text{fppf}} \quad \text{"Conti map" of fppf sites}$$

$$R\text{-Grp} = \text{Shv}_{\text{Ab}}^{\text{fppf}}(\text{Spec } R_0) \begin{array}{c} \xleftarrow{\varphi^{-1}} \\ \perp \\ \xrightarrow{\varphi_*} \end{array} \text{Shv}_{\text{Ab}}^{\text{fppf}}(\text{Spec } R) = R\text{-Grp}$$

unit is  $\eta_G: G \rightarrow \varphi_* \varphi^{-1} G : A \mapsto G(A/IA)$

so  $G_I = \text{ker } \eta_G$

Lemma  $G$ : (commutative) formal group /  $R$

$\Rightarrow N^\nu$  kills  $G_I$

proof choose coordinates Smooth connected

$$G \cong \text{Spf } R[x_1, \dots, x_n]$$

$$\text{group str: } R[x_1, \dots, x_n] \rightarrow R[x_1, \dots, x_n] \hat{\otimes} R[x_1, \dots, x_n]$$

$$x_i \mapsto f_i(\vec{x}, \vec{y})$$

$$\rightsquigarrow \begin{cases} \vec{f}(\vec{f}(\vec{x}, \vec{y}), \vec{z}) = \vec{f}(\vec{x}, \vec{f}(\vec{y}, \vec{z})) \\ \vec{f}(\vec{x}, 0) = \vec{x} = \vec{f}(0, \vec{x}) \\ \vec{f}(\vec{x}, \vec{y}) = \vec{f}(\vec{y}, \vec{x}) \end{cases}$$

$$\rightsquigarrow \vec{f}(\vec{x}, \vec{y}) = \vec{x} + \vec{y} + (\text{deg} \geq 2)$$

$$[N]\vec{x} := \vec{f}(\vec{x}, \dots, \vec{x}) = \cancel{N\vec{x}} + (\text{deg} \geq 2 \text{ of } x_1, \dots, x_n)$$

$N=0$  in  $R$

$$[N^\nu]\vec{x} = (\text{deg} \geq 2^\nu)$$

WTS:  $N^\nu = 0$  on  $G_I(A)$ ,  $\forall A \in \text{CRing}_R$

$$G_I(A) = \left\{ \begin{array}{c} f \mid R[\vec{x}] \xrightarrow{f} A \text{ conti} \\ \quad \quad \quad \searrow \circ \quad \downarrow \\ \quad \quad \quad \quad A/I_A \end{array} \right\}$$

$$f(x_i) \in IA$$

$$\leadsto N^\nu f(x_i) = f([N^\nu] x_i) \in I^{\nu+1} A = 0 \quad \square$$

$$\sim f\left(\sum x_i^{\deg \geq 2^\nu}\right) = \sum \underbrace{f(x_i)^{\deg \geq 2^\nu}}_{\in IA}$$

Cor  $G \in R\text{-Grp}$  s.t.  $\hat{G}$ : fpf locally covered by formal groups

$$\Rightarrow N^\nu \text{ kills } G_I$$

↑ formal groups

i.e.  $\exists \{R \rightarrow R_\alpha\}$  fpf cover.

$$\text{CRing}_R \xrightarrow{\hat{G}} \text{Ab}$$

$$\begin{array}{c} \uparrow \\ \text{CRing}_{R_\alpha} \end{array} \xrightarrow{\hat{G}_\alpha} \text{formal group}$$

proof

$$\bullet (G_I)_I \subset (\hat{G})_I \subset G_I \quad \leadsto (\hat{G})_I = G_I$$

$$\begin{array}{c} \parallel \\ G_I \end{array} \xrightarrow{\quad} G_I(A/IA) = \ker(G(A/IA) \xrightarrow{\cong} G(A/IA)/\langle I \cdot A/IA \rangle)$$

is zero 0

$$\bullet \hat{G}_I \text{ is covered by } (\hat{G}_\alpha)_I : \text{killed by } N^\nu.$$

$$\leadsto \hat{G}_I : \text{killed by } N^\nu \text{ (by the sheaf property)}$$

□

Lemma A Let  $G, H \in R\text{-Grp}$ , suppose

(a)  $G$  is  $N$ -divisible (i.e.,  $G \xrightarrow{N} G$  epi)

(b)  $\hat{H}$  is fpf-covered by formal groups

(c)  $H$  is formally smooth

(i.e.,  $J \subset A$  nilpotent  $\Rightarrow H(A) \rightarrow H(A/J)$ )

(Recall  $\varphi: R \rightarrow R/I = R_0$  "reduction mod  $I$ "  
 $\rightsquigarrow R_0\text{-Grp} \xleftarrow[\varphi_*]{\varphi^{-1}} R\text{-Grp}$   $\left[ \begin{array}{l} \varphi^{-1}: \text{res to } \mathcal{C}Alg_{R_0} \\ \varphi_* G_0 = G_0(- \otimes R_0) \end{array} \right.$

Then (Set  $G_0 = \varphi^{-1}G$ ,  $H_0 = \varphi^{-1}H$ )

(1)  $\text{Hom}_{R\text{-Grp}}(G, H)$ ,  $\text{Hom}_{R_0\text{-Grp}}(G_0, H_0)$

$\varphi_{G,H}^{-1} = \theta$

have no  $N$ -torsion

(2)  $\text{Hom}_{R\text{-Grp}}(G, H) \xrightarrow{\theta} \text{Hom}_{R_0\text{-Grp}}(G_0, H_0)$  : injective

(3)  $\forall f_0: G_0 \rightarrow H_0 \exists! g: G \rightarrow H$  s.t.  $\theta(g) = N^{\nu} f_0$

(4)  $f_0 \in \text{Im } \theta \iff g|_{G[N^{\nu}]} = 0$   
 $\uparrow$   
 in (3)





Proof (1)  $\varphi^{-1}$  preserves colim  $\Rightarrow$  preserves epimorphisms

$\leadsto N: G_0 \rightarrow G_0$  epi by (a)

$\leadsto N \subset \text{Hom}(G, H), \text{Hom}(G_0, H_0)$   
: mono.

$$(2) \text{Hom}(G, H) \xrightarrow{\theta} \text{Hom}(\varphi^{-1}G, \varphi^{-1}H)$$

$$\searrow \eta_H \quad \parallel \quad \text{Hom}(G, \varphi_*\varphi^{-1}H)$$

$$\text{Ker } \theta \cong \text{Hom}(G, \text{Ker } \eta_H) \subset N^\nu = 0$$

$\parallel$   
 $H_I$

mono by (a)  $\uparrow$   
 by Lem & (b)

So  $\text{Ker } \theta = 0$ .

(3) Uniqueness: by (2)  
explicitly construct  $g$

by (c)

$$0 \rightarrow H_I(A) \rightarrow H(A) \rightarrow H(A/I_A) \rightarrow 0$$

$$\leadsto 0 \rightarrow H_I \rightarrow H \xrightarrow{\varphi_*\varphi^{-1}} \varphi_*\varphi^{-1}H \rightarrow 0$$

$\eta_H$

$N^\nu$   
 $\downarrow$   
 $H$

$\downarrow$   
 $0$   
 by (b)

$\swarrow$   
 $S! \exists$

Define

$$\begin{array}{ccc}
 G & \xrightarrow{g} & H \\
 \eta_G \downarrow & \circlearrowright & \uparrow s \\
 \varphi_* \varphi^{-1} G & \xrightarrow{\varphi_* f_0} & \varphi_* \varphi^{-1} H \\
 \underbrace{\phantom{\varphi_* \varphi^{-1} G}}_{G_0} & & \underbrace{\phantom{\varphi_* \varphi^{-1} H}}_{H_0}
 \end{array}
 \xrightarrow{N^\nu} \varphi_* \varphi^{-1} H$$

$$\begin{array}{ccc}
 & \xrightarrow{\eta_H} & \\
 \eta_H \searrow & \circlearrowright & \swarrow \eta \\
 & \xrightarrow{\eta_H} &
 \end{array}$$

then

$$\begin{array}{ccc}
 \varphi^{-1} G & \xrightarrow{\varphi^{-1} g = \theta(g)} & \varphi^{-1} H \\
 \downarrow \varphi^{-1} \eta_G & \circlearrowright & \downarrow \varphi^{-1} \eta_H \\
 \varphi^{-1} \varphi_* \varphi^{-1} G & \xrightarrow{\varphi^{-1} \varphi_* N^\nu f_0} & \varphi^{-1} \varphi_* \varphi^{-1} H \\
 \Sigma_{\varphi^{-1} G} \downarrow & \circlearrowright & \downarrow \Sigma_{\varphi^{-1} H} \\
 \varphi^{-1} G & \xrightarrow{N^\nu f_0} & \varphi^{-1} H
 \end{array}$$

so  $\theta(g) = N^\nu f_0$ .

(A)  $\theta : \text{inj} \rightsquigarrow \theta(f) = f_0 \iff N^\nu f = g$

$$0 \rightarrow G[N^\nu] \rightarrow G \xrightarrow{N^\nu} G \rightarrow 0$$

$$\begin{array}{ccc}
 & g \downarrow & \\
 0 & \searrow & H \\
 & & \swarrow f
 \end{array}$$

$\rightsquigarrow \exists f \iff g|_{G[N^\nu]} = 0$

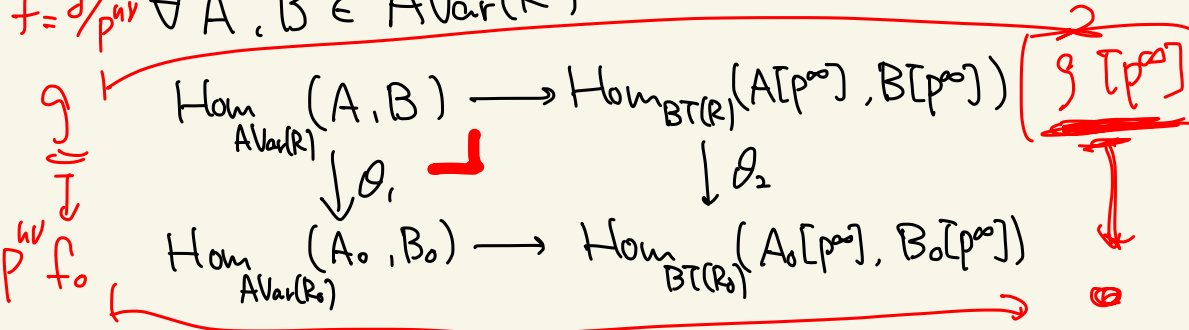


# §1.2 Proof of the theorem

$$N = p^n$$

Step 1 show  $\star$  is fully faithful, i.e.

$$f = g/p^{nv} \forall A, B \in \text{AVar}(R)$$



ab. sch. & p-div gps satisfy (a)-(c) of Lemma A

$\rightarrow \theta_1, \theta_2$  : injective.

WTS  $\forall f_0 : A_0 \rightarrow B_0$

$$[ \underline{f_0[p^\infty]} \in \text{Im } \theta_2 \Rightarrow f_0 \in \text{Im } \theta_1 ]$$

$\exists! g : A \rightarrow B$  s.t.  $g = p^{nv} f_0$ .

Need  $g|_{A[p^{nv}]} = 0$ .

but  $g[p^\infty] : A[p^\infty] \rightarrow B[p^\infty]$  lifts  $p^{nv} \underline{f_0[p^\infty]} \in \text{Im } \theta_2$

$$\Rightarrow \underline{g|_{A[p^{nv}]} = g[p^\infty]|_{A[p^{nv}]} = 0}$$



## Step 2 essential surjectivity

$$\begin{array}{ccc}
 \text{AVar}(R) & \longrightarrow & \text{BT}(R) \ni G \\
 \downarrow B & & \downarrow \\
 \text{AVar}(R_0) & \longrightarrow & \text{BT}(R_0) \ni G_0 \\
 \downarrow \psi & \longleftarrow & \downarrow \varepsilon \\
 A_0 & \longleftarrow & A_0[p^\infty]
 \end{array}$$

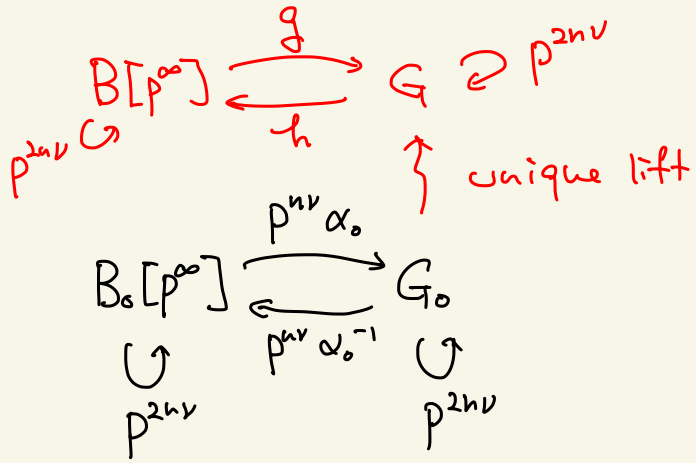
Fact  $\text{AVar}(R) \longrightarrow \text{AVar}(R_0)$ : *ess. surj.*

Idea • lifting of the underlying scheme exists by smoothness.  $f: A \rightarrow \text{Spec } R_0$   
 (obstruction  $\in \text{Ext}^2(L_{A/R_0}, f^*I) = 0$ )

- prove: abelian group str. lifts uniquely by deforming the graph of the structure maps

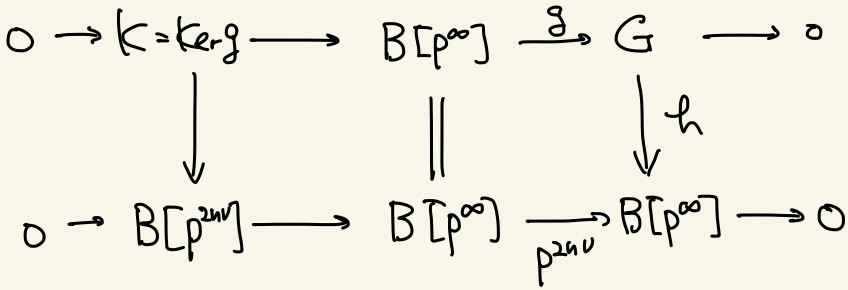
$$\begin{array}{c}
 \rightsquigarrow \exists B \\
 \downarrow \\
 B_0 \xrightarrow[\alpha_0]{\sim} A_0 \rightsquigarrow B_0[p^\infty] \xrightarrow[\alpha_0]{\sim} A_0[p^\infty] \rightsquigarrow G_0
 \end{array}$$

BT(R)  
 $\downarrow$   
 BT(R<sub>0</sub>)

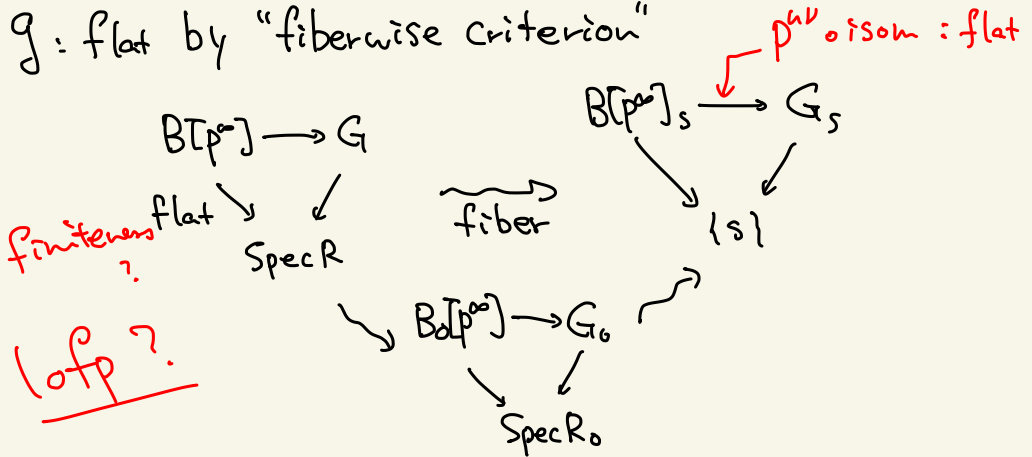


$\rightsquigarrow g, h$ : isogeny

Consider



$g$ : flat by "fiberwise criterion"



$\leadsto K \hookrightarrow B[p^{2\nu}]$  finite flat subgroup

$$A = B/K \in \text{AVar}(R)$$

Then

AVar

BT

R

$$K \rightarrow B \rightarrow A$$

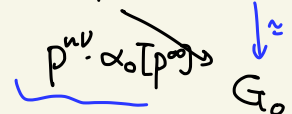
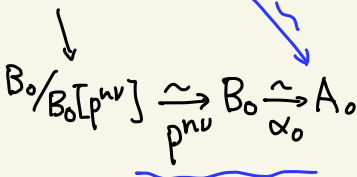
$$K \rightarrow B[p^\infty] \rightarrow A[p^\infty]$$



$R_0$

$$B_0[p^{2\nu}] \rightarrow B_0 \rightarrow A \otimes_R R_0$$

$$B_0[N^\nu] \rightarrow B_0[p^\infty] \rightarrow A_0[p^\infty]$$



↑ compatible

i.e.  $A$  lifts  $(A_0, G, \alpha : A_0[p^\infty] \rightarrow G_0)$



# §2 Serre-Tate coordinates for deformations of ordinary abelian varieties

## Notations

char  $p$

- $A \in \text{AVar}(k)$ .  $k$ : field,  $g = \dim A$   
(cartier)
- $(-)^t$ : the dual of ab sch / fin flat gp /  $p$ -div gp
- $T_p A(k) := \varprojlim_n A(k)[p^n]$  : the Tate module  
(a  $\mathbb{Z}_p$ -module)
- $k$ : field  $\rightsquigarrow \text{Art}_k$ : the cat of augmented  
 $(R, \mathfrak{m})$  artin local  $k$ -alg

$$\begin{array}{ccc} \widehat{\mathcal{M}}_{A/k} : \text{Art}_k & \longrightarrow & \text{Cat} \xrightarrow{\pi_0} \text{Set} \\ \cup & & \cup \\ \mathcal{R} & \longmapsto & \{A\} \times_{\text{AVar}(k)} \text{AVar}(R) \end{array}$$

"the formal moduli space"

known to be prorepresentable by  $\text{Spf } \mathcal{R}$ ,

$$\mathcal{R} \simeq W(k)[[t_{ij} \mid 1 \leq i, j \leq g]]$$

Thm Setting:  $k$ : alg closed field  $\supset \mathbb{F}_p$

$A, B \in \text{AVar}(k)$  ordinary

(1)  $\exists$  natural isom  $\widehat{G}_m^{\otimes 2} \cong \widehat{G}_m$

$$\widehat{M}_{A/k}(R) \xrightarrow{\cong} \text{Hom}_{\mathbb{Z}_p}(\underbrace{T_p A(k)}_{\psi} \otimes_{\mathbb{Z}_p} \underbrace{T_p A^t(k)}_{\psi}, \widehat{G}_m(R))$$

$$\underbrace{A/R}_{\psi} \mapsto \underbrace{\mathcal{G}(A/R; -, -)}_{\psi}$$

of functors  $\text{Art}_k \rightarrow \text{Set}$

(2) Compatible with the duality:

$$\widehat{M}_{A/k}(R) \xrightarrow{\cong} \text{Hom}_{\mathbb{Z}_p}(T_p A(k) \otimes_{\mathbb{Z}_p} T_p A^t(k), \widehat{G}_m(R))$$

$$\begin{array}{ccc} (-)^t \downarrow & \curvearrowright & \downarrow \cong \\ \widehat{M}_{A^t/k}(R) & \xrightarrow{\cong} & \text{Hom}_{\mathbb{Z}_p}(T_p A^t(k) \otimes_{\mathbb{Z}_p} T_p A^{tt}(k), \widehat{G}_m(R)) \end{array}$$

(3)  $A, B$ : lifts of  $A, B$ , then  $A \xrightarrow{f} B$  lifts to  $\exists A \xrightarrow{f} B$

iff

$$\begin{array}{ccc} \text{id} \otimes f^t \nearrow & T_p A(k) \otimes_{\mathbb{Z}_p} T_p A^t(k) & \searrow \mathcal{G}(A/R; -, -) \\ & & \widehat{G}_m(R) \\ T_p A(k) \otimes_{\mathbb{Z}_p} T_p B^t(k) & \curvearrowright & \\ f \otimes \text{id} \searrow & T_p B(k) \otimes_{\mathbb{Z}_p} T_p B^t(k) & \nearrow \mathcal{G}(B/R; -, -) \end{array}$$



# Some definitions & preliminaries (R:artin local)

•  $X$ : fin flat  $R$ -group scheme is

• étale if  $X \rightarrow \text{Spec } R$  : étale

dual ↷

• of multiplicative type if étale-locally (on  $\text{Spec } R$ )

isomorphic to  $\exists \bigoplus_i \mu_{n_i}$

•  $X$ :  $p$ -divisible group /  $R$  is

• étale if  $X[p^n]$ : étale  $\forall n$

Connected-étale seq

$$0 \rightarrow X_{\text{conn}} \rightarrow X \rightarrow X_{\text{ét}} \rightarrow 0$$

Connected  $\nearrow$   
component of the unit

$\uparrow$  maximal étale quotient

← canonically splits if  $R$ : perfect field

• Connected if  $X_{\text{ét}} = 0$

• toric (toroidal) if  $X[p^n]$ : multiplicative type

$\Leftrightarrow X^t$ : étale

$$0 \rightarrow X^t_{\text{conn}} \rightarrow X^t \rightarrow X^t_{\text{ét}} \rightarrow 0$$

dual ↷

$$0 \rightarrow (X^t_{\text{ét}})^t \rightarrow X \rightarrow (X^t_{\text{conn}})^t \rightarrow 0$$

$= X_{\text{tor}}$

$$X_{\text{tor}} \hookrightarrow X_{\text{conn}}.$$

these are representable by formal groups /  $R$

• ordinary if  $X_{\text{tor}} \xrightarrow{\cong} X_{\text{conn}}$ , i.e.

$$0 \rightarrow X_{\text{tor}} \rightarrow X \rightarrow X_{\text{ét}} \rightarrow 0$$

•  $A$ : abelian variety

$\leadsto A[p^\infty]_{\text{conn}} \cong \hat{A}$ : the formal completion  
along the zero section  
 $\text{Spec } R \rightarrow A$

$$\left( \hat{A}(R) = \ker(A(R) \rightarrow A(R^{\text{red}})) \text{ for } R: \text{artinian local} \right)$$

•  $A$  is ordinary if  $A[p^\infty]$  is ordinary

$$\Leftrightarrow A[p^n] \cong (\mathbb{Z}/p^n\mathbb{Z})^g$$

$$\Leftrightarrow T_p A(k) \cong \mathbb{Z}_p^g$$

typical examples

fin flat X

$X$	unip	mult
$X/x^t$	Conn	$\hat{\mathcal{E}}t$
Conn	$\alpha_p$	$\mu_{p^n}$
$\hat{\mathcal{E}}t$	$\mathbb{Z}/p^n\mathbb{Z}$	$\mathbb{Z}/2$

$$\alpha_p = \ker(\text{Fr}: G_a \rightarrow G_a)$$

$$\mu_{p^n} \rightsquigarrow \mu_{p^\infty}$$

$\nwarrow$  dual

$$\begin{aligned} &= \\ &\frac{1}{p^n}\mathbb{Z}/\mathbb{Z} \end{aligned}$$

}

$$\underline{\mathbb{Q}_p/\mathbb{Z}_p}$$

# Construction / Proof of (1)

$$A \in \pi_0 \left( \{A\} \times_{A\text{Var}(k)} A\text{Var}(R) \right)$$

$$\downarrow \cong \downarrow \text{ by } \S 1$$

$$A[p^\infty] \in \pi_0 \left( \{A[p^\infty]\} \times_{BT(k)} BT(R) \right)$$

$$\downarrow \cong \downarrow \text{ (1)}$$

$$0 \rightarrow \boxed{\text{tor}} \rightarrow A[p^\infty] \rightarrow \boxed{\hat{e}t} \rightarrow 0$$

$$\text{Ext}_{R\text{-Grp}}^1 \left( \frac{T_p A(k) \otimes \mathbb{Q}_p / \mathbb{Z}_p}{\boxed{\hat{e}t}}, \text{Hom}_{\mathbb{Z}_p} (T_p A^t(k), \hat{G}_m) \right)$$

$$\boxed{\text{tor}} \rightarrow A[p^\infty]$$

$$\psi_A \uparrow \quad \quad \quad \uparrow$$

$$T_p A(k) \rightarrow T_p A(k) \otimes \mathbb{Q}_p$$

$$\psi_{A/R} \hookrightarrow$$

$$\begin{array}{c} \uparrow \cong \downarrow \text{ (2)} \\ \text{Hom}_{R\text{-Grp}} \left( T_p A(k), \text{Hom}_{\mathbb{Z}_p} (T_p A^t(k), \hat{G}_m) \right) \end{array}$$

$\cong \downarrow$  R-points & tensor-hom

$$\text{Hom}_{\mathbb{Z}_p} (T_p A(k) \otimes T_p A^t(k), \hat{G}_m(R))$$

① Classifying the lift  $A[p^\infty]_{\mathbb{R}}$  of  $A[p^\infty]_{\mathbb{K}}$   
 $n \gg 0$  so that  $p^n$  kills  $\hat{A}$

$$\begin{array}{ccccccc}
 \rightsquigarrow \textcircled{\oplus} & 0 & \longrightarrow & \hat{A} & \longrightarrow & A[p^n] & \longrightarrow & A(\mathbb{K})[p^n] \\
 & & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & \longrightarrow & \hat{A} & \longrightarrow & A & \longrightarrow & A(\mathbb{K}) \longrightarrow 0 \\
 & & & \downarrow 0 & & \downarrow p^n & & \downarrow \\
 & 0 & \longrightarrow & \hat{A} & \longrightarrow & A & \longrightarrow & A(\mathbb{K}) \longrightarrow 0 \\
 & & & \downarrow & & & & \\
 & & & \hat{A} & & & & 
 \end{array}$$

vary  $n$

$$\begin{array}{ccccccc}
 \rightsquigarrow & 0 & \longrightarrow & \hat{A} & \longrightarrow & A[p^n] & \longrightarrow & A(\mathbb{K})[p^n] & \longrightarrow & \hat{A} \\
 & & & \text{id} \downarrow & & \downarrow & & \downarrow & & \downarrow p \\
 & 0 & \longrightarrow & \hat{A} & \longrightarrow & A[p^{n+1}] & \longrightarrow & A(\mathbb{K})[p^{n+1}] & \longrightarrow & \hat{A}
 \end{array}$$

} column

$$0 \longrightarrow \hat{A} \longrightarrow A[p^\infty] \longrightarrow T_p A(\mathbb{K}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

this reduces to the (canonically split)  
 coun-ét seq /  $\mathbb{K}$  :

$$0 \longrightarrow \hat{A} \longrightarrow A[p^\infty] \longrightarrow T_p A(\mathbb{K}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0$$

$$A: \text{ordinary} \iff \hat{A} \simeq ((A[\mathfrak{p}^\infty]^t)_{\text{ét}})^t$$

$$\rightsquigarrow \begin{cases} \hat{A}[\mathfrak{p}^n] \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}_p}(A^t(k)[\mathfrak{p}^n], \mathcal{M}_{\mathfrak{p}^n}) \\ \hat{A} \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}_p}(T_p A^t(k), \hat{G}_m) \end{cases}$$

did I use  
also closed?  
?

note:  $\hat{G}_m(R) \simeq 1+m \simeq \mu_{\text{p-ad}}(R)$

f.f. groups of multiplicative type / toroidal p-div  
gps /  $k$

lifts uniquely to  $R$

(probably b/c taking the dual, by  $\hat{E}_t/k \simeq \hat{E}_t/R$  ?)

$$\rightsquigarrow \begin{cases} \hat{A}[\mathfrak{p}^n] \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}_p}(A^t(k)[\mathfrak{p}^n], \mathcal{M}_{\mathfrak{p}^n}) \\ \hat{A} \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}_p}(T_p A^t(k), \hat{G}_m) \end{cases}$$

$$\rightsquigarrow 0 \rightarrow \hat{A} \rightarrow A[\mathfrak{p}^\infty] \rightarrow T_p A(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0$$

pre-determined

$$\begin{array}{ccccccc} & & & \downarrow & & & \\ 0 & \rightarrow & \hat{A} & \rightarrow & A[\mathfrak{p}^\infty] & \rightarrow & T_p A(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0 \\ & & \text{foric} & & & & \text{étale} \end{array}$$

classifying  $A[\mathfrak{p}^\infty] \iff$  classifying ext

② In general

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \quad \text{in } \mathcal{A} \text{ abelian}$$

$(X \rightarrow Y \rightarrow Z \rightarrow 0)$  distinguished triangle in  $D(\mathcal{A})$

$$\rightsquigarrow \text{Hom}(Y, \mathcal{D}) \xrightarrow{\textcircled{a}} \text{Hom}(X, \mathcal{D}) \xrightarrow{\textcircled{b}} \text{Ext}^1(Z, \mathcal{D}) \xrightarrow{\textcircled{c}} \text{Ext}^1(Y, \mathcal{D})$$

• If  $\textcircled{a}, \textcircled{c} = 0$ , then  $\textcircled{b}$  bijective

explicitly

$$\Omega Y \rightarrow \Omega Z \rightarrow X \rightarrow Y : \text{fiber seq.}$$

$$\text{Ext}^1(Z, \mathcal{D}) \ni \begin{matrix} \searrow \\ \mathcal{D} \end{matrix} \stackrel{\exists!}{\rightarrow} \text{Hom}(X, \mathcal{D})$$

$$\begin{array}{ccccccc} \Omega Z & \rightarrow & X & \rightarrow & Y & \rightarrow & Z \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ \Omega Z & \xrightarrow{f} & \mathcal{D} & \rightarrow & \text{cofib}(f) & \rightarrow & Z \end{array}$$

Our case:  $\mathcal{A} = R\text{-Grp}$

$$0 \rightarrow \underset{X}{T_p A(k)} \rightarrow \underset{Y}{T_p A(k) \otimes \mathbb{Q}_p} \rightarrow \underset{Z}{T_p A(k) \otimes \mathbb{Q}_p / \mathbb{Z}_p} \rightarrow 0$$

$$\mathcal{D} = \text{Hom}_{\mathbb{Z}_p}(T_p A^e(k), \hat{G}_m) \simeq \hat{A}$$

•  $\text{Hom}_{R\text{-Grp}}(T_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \hat{A}) = 0 \Rightarrow \textcircled{a} = 0$

mult. by p is  
surjective

formal group  
→ killed by p-power

•  $\text{Ext}'_{R\text{-Grp}}(T_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \hat{A}) \rightarrow \text{Ext}'_{R\text{-Grp}}(T_p A(k), \hat{A})$  injective  
⇒  $\textcircled{a} = 0$

Sketch  $T_p A(k) \cong \mathbb{Z}_p^{\oplus g}$ ,  $\hat{A} \cong \mu_{p^\infty}^{\oplus g}$  → enough to show

$\text{Ext}'_{R\text{-Grp}}(\mathbb{Q}_p, \mu_{p^\infty}) \rightarrow \text{Ext}'_{R\text{-Grp}}(\mathbb{Z}_p, \mu_{p^\infty}) : \text{inj}$

$\lim_{\leftarrow} \mathbb{Z}_p \searrow \textcircled{1}$

$\textcircled{2} \nearrow \text{pr}$

$\lim_{\leftarrow} \text{Ext}'_{R\text{-Grp}}(\mathbb{Z}_p, \mu_{p^\infty})$   
↑ mult by p

$\mathbb{Z}_p$	$\xrightarrow{1}$	$\mathbb{Z}_p$	$\rightarrow 0$
$\parallel$		$\downarrow p$	$\downarrow$
$\mathbb{Z}_p$	$\xrightarrow{p}$	$\mathbb{Z}_p$	$\rightarrow \mathbb{Z}/p$
$\parallel$		$\downarrow p$	$\downarrow$
$\mathbb{Z}_p$	$\xrightarrow{p^2}$	$\mathbb{Z}_p$	$\rightarrow \mathbb{Z}/p^2$
$\vdots$		$\vdots$	$\vdots$
$\mathbb{Z}_p$		$\mathbb{Q}_p$	$\rightarrow \mathbb{Q}_p/\mathbb{Z}_p$

$\textcircled{1} : \text{inj}$

$\pi_1 \lim_{\leftarrow} \text{Map}(\mathbb{Z}_p, \mu_{p^\infty}) \rightarrow \lim_{\leftarrow} \pi_{-1} \text{Map}(\mathbb{Z}_p, \mu_{p^\infty})$

$\text{Ker} = \lim_{\leftarrow}^1 \pi_0 \text{Map}(\mathbb{Z}_p, \mu_{p^\infty})$

Mittag-Leffler condition ✓

$\textcircled{2} : \text{inj}$

follows from  $H'_{\text{ét}}(\text{Spec } R, \mu_{p^\infty}) = 0$

↪ not sure how



$$\begin{array}{ccc}
 \text{So } \Omega(T_p A(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p) & \xrightarrow{A[p^\infty]} & \text{Hom}_{\mathbb{Z}_p}(T_p A^t(k), \widehat{G}_m) \\
 \downarrow & \nearrow \exists! \varphi_{A/R} & \\
 T_p A(k) & & 
 \end{array}$$

$\sim$   
 $[-1]$

$$0 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Q}_p \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0 \quad \left( (1) \square \right)$$

$\otimes T_p A(k)$

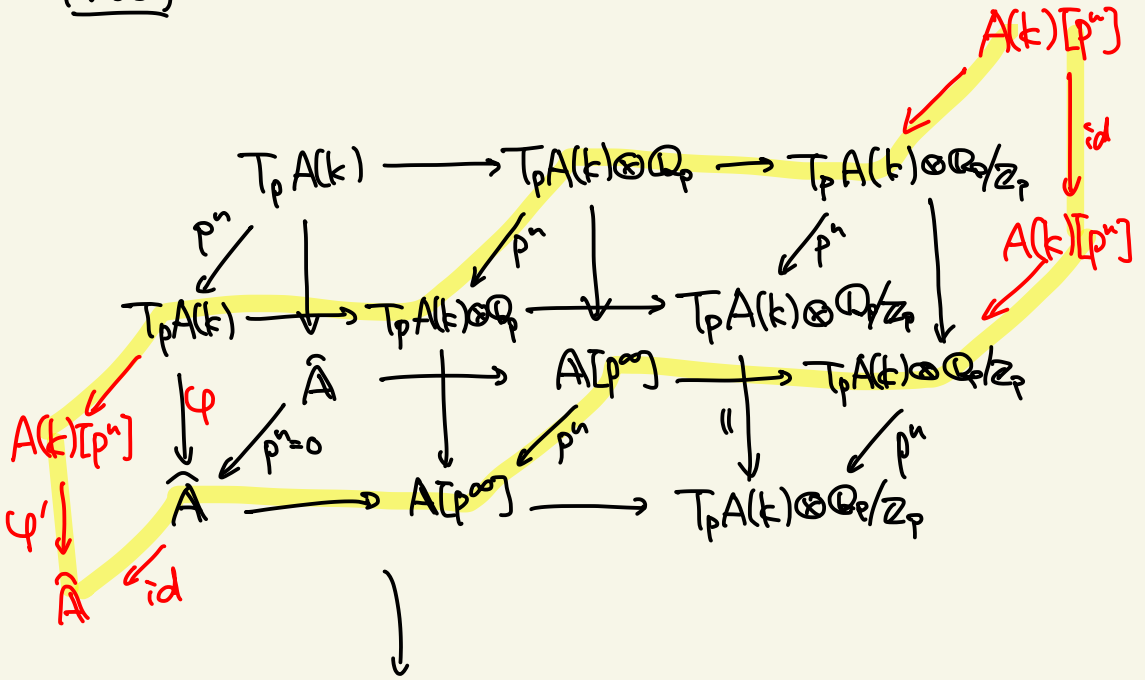
Remark

More explicitly,  $\varphi_{A/R}$  is given as follows:

take  $n \gg 0$  so that  $m^{n+1} = 0$   
 (p.e.m., so  $p^n$  kills  $\widehat{A}$ )

$$\begin{array}{ccccccc}
 & & & & T_p A(k) & & \\
 & & & & \downarrow & & \\
 \widehat{A} & \rightarrow & A[p^n] & \rightarrow & A(k)[p^n] & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 \rightarrow \widehat{A} & \rightarrow & A & \rightarrow & A(k) & \rightarrow & 0 \\
 \downarrow 0 & & \downarrow p^n & & \downarrow p^n & & \\
 0 \rightarrow \widehat{A} & \rightarrow & A & \rightarrow & A(k) & & \\
 \downarrow & & & & & & \\
 \widehat{A} & & & & & & 
 \end{array}$$

# Proof



$$A(k)[p^n] \xleftarrow{id} A(k)[p^n]$$

$$\begin{array}{ccc} \varphi' \downarrow & & \downarrow id \\ \hat{A} & \longleftarrow & A(k)[p^n] \end{array}$$

i.e.,  $\varphi' = (\text{the connecting hom})$

# proof of (3)

$$\begin{array}{ccc}
 A \overset{\text{---}}{\dashrightarrow} B & & A[p^\infty] \overset{f[p^\infty]}{\text{---}} B[p^\infty] \\
 \downarrow & \text{Serre-Tate} & \downarrow \\
 A \xrightarrow{f} B & \longleftarrow & A[p^\infty] \xrightarrow{f[p^\infty]} B[p^\infty]
 \end{array}$$

by construction of (1),  $f[p^\infty]$  is a morphism filling

$$\begin{array}{ccccc}
 \text{Hom}_{\mathbb{Z}_p}(T_p A^t(k), \hat{G}_m) & \rightarrow & A[p^\infty] & \rightarrow & T_p A(k) \otimes \mathbb{Q}_p / \mathbb{Z}_p \\
 \text{Hom}(p^t, \text{id}) \downarrow & & \downarrow f[p^\infty] & & \downarrow f \otimes \text{id} \\
 \text{Hom}_{\mathbb{Z}_p}(T_p B^t(k), \hat{G}_m) & \rightarrow & B[p^\infty] & \rightarrow & T_p B(k) \otimes \mathbb{Q}_p / \mathbb{Z}_p
 \end{array}$$

pre-determined by the unique liftability of toroidal / étale groups to nilpotent thickening

rotating the triangle :

$$\begin{array}{ccccc}
 \Omega(T_p A(k) \otimes \mathbb{Q}/\mathbb{Z}_p) & \xrightarrow{\quad} & \text{Hom}_{\mathbb{Z}_p}(T_p A^t(k), \hat{G}_m) & \rightarrow & A[\mathbb{P}^\infty] \\
 \downarrow f \otimes \text{id} & \text{AT}[\mathbb{P}^\infty] & \text{Hom}(f^t, \text{id}) \downarrow & & \downarrow f[\mathbb{P}^\infty] \\
 \Omega(T_p B(k) \otimes \mathbb{Q}/\mathbb{Z}_p) & \xrightarrow{\quad} & \text{Hom}_{\mathbb{Z}_p}(T_p B^t(k), \hat{G}_m) & \rightarrow & B[\mathbb{P}^\infty] \\
 & \text{BT}[\mathbb{P}^\infty] & & & 
 \end{array}$$

$\exists f[\mathbb{P}^\infty] \iff$  left square commutes

$$\begin{array}{ccccc}
 & \nearrow T_p A(k) & \xrightarrow{\exists! \varphi_{A/R}} & & \\
 \Omega(T_p A(k) \otimes \mathbb{Q}/\mathbb{Z}_p) & \xrightarrow{\quad} & \text{Hom}_{\mathbb{Z}_p}(T_p A^t(k), \hat{G}_m) & \rightarrow & A[\mathbb{P}^\infty] \\
 \downarrow f \otimes \text{id} & \downarrow f & \text{Hom}(f^t, \text{id}) \downarrow & & \downarrow f[\mathbb{P}^\infty] \\
 \Omega(T_p B(k) \otimes \mathbb{Q}/\mathbb{Z}_p) & \xrightarrow{\quad} & \text{Hom}_{\mathbb{Z}_p}(T_p B^t(k), \hat{G}_m) & \rightarrow & B[\mathbb{P}^\infty] \\
 & \searrow T_p B(k) & \xrightarrow{\exists! \varphi_{B/R}} & & 
 \end{array}$$

$\implies$  blue square commutes

$$\begin{array}{ccc}
 \text{id} \otimes \text{id} \nearrow & T_p A(k) \otimes T_p A^t(k) & \xrightarrow{\varphi(A/R; -, -)} \\
 & \downarrow & \downarrow \\
 T_p A(k) \otimes T_p B^t(k) & \xrightarrow{\quad} & \hat{G}_m \\
 \text{id} \otimes f^t \searrow & T_p B(k) \otimes T_p B^t(k) & \xrightarrow{\varphi(B/R; -, -)}
 \end{array}$$

Proof of (2) is technical and unenlightening  
contrary to its formal appearance ...

# § 3

Canonically an  
 $W(k)$ -alg  
 $\downarrow$   
 $(\mathcal{R}, \mathfrak{m})$

$$\hat{\mathcal{M}}_{A/k} : \text{Art}_k \rightarrow \text{Set}$$

representable by a complete local alg  $(\mathcal{R}, \mathfrak{m})$

$$\hat{\mathcal{M}}_{A/k} \simeq \text{Spf } \mathcal{R} \rightsquigarrow \text{formal } W(k)\text{-group}$$

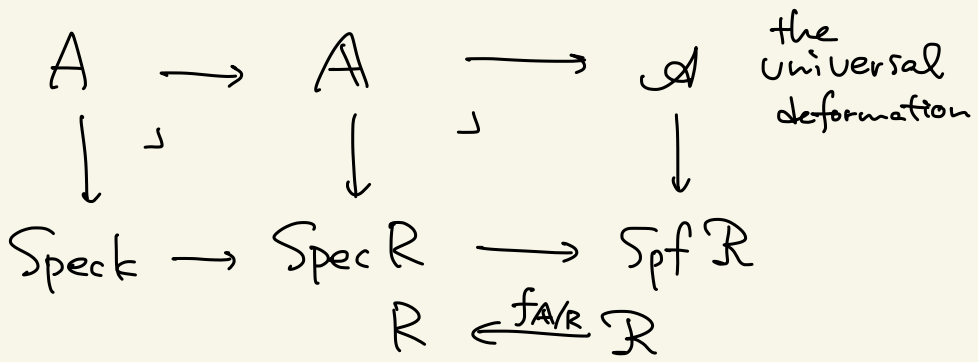
(e.g. by Schlessinger's representability thm)

$$\left( \begin{array}{l} \text{Since } \hat{\mathcal{M}}_{A/k} \xrightarrow[\mathfrak{q}]{\sim} \text{Hom}_{\mathbb{Z}_p} (T_p A(k) \otimes T_p A^t(k), \hat{\mathbb{G}}_m) \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \hat{\mathbb{G}}_m^{\mathfrak{g}^2} \\ \text{by picking } \mathbb{Z}_p\text{-basis } \alpha_1, \dots, \alpha_g \text{ of } T_p A \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \beta_1, \dots, \beta_g \text{ of } T_p A^t \\ \text{and setting} \\ T_{ij} = \mathfrak{q}(\alpha_i, \alpha_j) - 1 \in \mathcal{R} \\ \rightsquigarrow W(k)[[T_{ij}]] \xrightarrow{\sim} \mathcal{R} \end{array} \right)$$

passing to the limit

$$\hat{\mathcal{M}}_{A/k} \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}_p} (T_p A(k) \otimes T_p A^t(k), \hat{\mathbb{G}}_m)$$

$$\mathfrak{q} : T_p A(k) \otimes T_p A^t(k) \xrightarrow[\cong]{\sim} \text{Hom}_{W(k)\text{-Grp}} (\hat{\mathcal{M}}, \hat{\mathbb{G}}_m)$$



$$\mathfrak{g}(A/R, \alpha, \beta) \longleftarrow \mathfrak{g}(\alpha, \beta)$$

$$KS: \underline{W}_{A/R} \rightarrow \text{Lie}(A^t/R) \otimes_R \Omega_{R/W}^1$$

$$\xrightarrow{\text{limit}} KS: \underline{W}_{A/R} \rightarrow \text{Lie}(A^t/R) \otimes_R \Omega_{R/W}^1$$

↖ Conti 1-forms

## The main theorem

$$\begin{array}{ccc}
 (\alpha, \beta) \in T_p A^{tt}(k) \otimes T_p A(k) & \xrightarrow{\mathfrak{g}} & \text{Hom}_{W\text{-Grp}}(\hat{U}, \hat{E}_m) \\
 \downarrow & & \downarrow d\log \\
 (w(\alpha), w(\beta)) \in \underline{W}_{A^t/R} \otimes \underline{W}_{A/R} & \curvearrowright & \\
 \downarrow \text{id} \times KS & & \\
 \underline{W}_{A^t/R} \otimes \text{Lie}(A^t/R) \otimes \Omega_{R/W}^1 & \xrightarrow{\text{pairing} \otimes \text{id}} & \Omega_{R/W}^1
 \end{array}$$

where  $w(\beta)$  is (the limit of)

$$\begin{array}{ccc}
 T_p A^t(k) & \xrightarrow{\sim} & \text{Hom}_{R\text{-Grp}}(\hat{A}, \hat{E}_m) \\
 & & \downarrow \text{Lie} \\
 & & \text{Hom}_{R\text{-Grp}}(\text{Lie}(A/R), \hat{E}_a) \\
 & \searrow w & \parallel \\
 & & \underline{W}_{A/R}
 \end{array}$$



## Restatement (§4)

Hodge - de Rham

$$0 \rightarrow \underline{W}_{\mathcal{A}/\mathcal{R}} \rightarrow H'_{dR}(\mathcal{A}/\mathcal{R}) \rightarrow \text{Lie}(\mathcal{A}^t/\mathcal{R}) \rightarrow 0$$

Gauss - Manin connection

$$\nabla : H_{\mathcal{A}} \rightarrow H_{\mathcal{A}} \otimes \Omega_{\mathcal{R}/W(k)}^1$$